A NEARLY PSEUDOCOMPACT SPACE

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ABSTRACT—Though infinitely many examples of nearly pseudocompact, non-pseudocompact spaces are known to exist, their existence is known solely as a logical consequence of several technical theorems. A detailed construction of a nearly pseudocompact, non-pseudocompact space has yet to appear in the literature. It is the objective of this paper to offer just such a construction.

Nearly pseudocompact spaces were first introduced by Henriksen and Rayburn (1980). To confirm that they had formulated a valid generalization of pseudocompactness, they produced an example of a topological space that was nearly pseudocompact but not pseudocompact. Unfortunately, the example's existence and its nearly pseudocompactness were only asserted as a logical consequence of several technical theorems. No detailed construction was offered. That detailed construction is precisely what is offered here. By actually going through the construction of such a space, the reader will perhaps gain a better idea of what a nearly pseudocompact space looks like, and indeed be able to see how many other such examples can be constructed. The paper will end with a look at two theorems of Hausdorff which will assure us that many such examples exist.

All of the examples to be presented here are linearly ordered spaces. The reader in need of a review of order relations, cardinals, and ordinals may find Willard (1970) helpful. It will be necessary to cite a definition and theorem in which a distinction is made between "measurable" and "non-measurable" cardinals. All of the cardinals used in this paper are of the "non-measurable" variety, thereby making the distinction moot. The reader who wishes to look into this distinction more deeply might refer to Gillman and Jerison (1960).

Certain concepts connected with linearly ordered spaces are worth reviewing. Two elements x and y of the linearly ordered space X are called consecutive, if there exists no element z of X with $x < z < y$. We say that a linearly ordered set is dense if no two elements are consecutive. A useful way of constructing new linearly ordered sets from two existing linearly ordered sets A and B is by simple concatenation. We denote this new set $A + B$, and agree that if $x \in A$ and $y \in B$, then $x < y$ in $A + B$. If X is a linearly ordered set we let X^* denote the same set with the inverse order.

EXAMPLE 1. $\omega_0 + \omega_2$ and $\omega_1 + 1 + \omega_0^*$ are linearly ordered spaces.

A cut of X is any pair (A, B) of subsets of a linearly ordered set X such that $A \cup B = X$, and whenever $x \in A$ and $y \in B$ then $x < y$. If A has no largest element and B has no smallest element, the cut is called a gap. If (A, B) is a cut and A or B is empty, it is a left- or right-end gap, respectively. It will be useful to think of gaps as being "virtual" elements, i.e. an element x such that $a < x < b$ for every $a \in A$ and every $b \in B$.

EXAMPLE 2. If $X = \omega_1 + \omega_0^*$, then (ω_1, ω_0^*) is a cut of X, and in particular, a gap of X. On the other hand, no interior cut of $Y = \omega_1 + 1 + \omega_0^*$ yields a gap.

If X is a linearly ordered set with at least two elements, then the set of all intervals of X form a basis for a topology on X. Every linearly ordered topological space is known to be hereditarily normal (Porter and Woods, 1988).

Recall that a set Z is called a zero-set of X if and only if $Z = f^{\leftarrow}(0)$ for some continuous function f on X. A set is called a cozero-set if it is the complement of a zero-set. A topological space X is called pseudocompact if and only if every continuous function on X is bounded, and X is called realcompact if and only if it is homeomorphic to a closed subset of a product of real lines. Perhaps less familiar to a reader with a background in topology are the following useful notions.

DEFINITION 1. Let X be normal. A set A is said to be relatively pseudocompact in X if and only if every continuous function of X is bounded on A. A set A is said to be relatively realcompact in X if and only if it is a subset of a closed realcompact subset of X.

Though the normality of X is not strictly speaking necessary to define the concepts of relatively real- and relatively pseudocompactness, a more general definition of relatively realcompactness would require a distracting review of the Hewitt-Nachbin realcompactification υX (see Schommer, 1993 for a general definition of relative realcompactness). Indeed, the classic definitions of nearly real- and nearly pseudocompact spaces rely on the reader's familiarity with υX as well as the Stone-Cêch compactification βX. Again, to avoid digressing into a review of these constructions, the following statements (theorems in the literature) will be adopted as our definitions of these topological properties.

DEFINITION 2. X is nearly realcompact if and only if every relatively pseudocompact cozero set is realcompact. X is nearly pseudocompact if and only if every realcompact cozero set is relatively pseudocompact.

Cozero-sets are known to inherit realcompactness (Gillman and Jerison, 1960). Thus realcompact spaces are nearly realcompact and pseudocompact spaces are nearly pseudocompact. Perhaps less obvious without a discussion of βX and υX are the following results:

Theorem 1. *The following statements are true.*

1. A realcompact, pseudocompact space is compact (Gillman and Jerison, 1960)

2. A nearly realcompact, pseudocompact space is compact (Blair and Van Douwen, 1992)

3. A realcompact, nearly pseudocompact space is compact (Henriksen and Reyburn, 1980)

In addition to being nearly pseudocompact and non-pseudocompact, the space we are about to construct will also prove to be nearly realcompact; an interesting result given Theorem 1.

A Nearly Pseudocompact, Non-Pseudocompact Space

Let X be a linearly ordered space and let X^+ be the union of X with all its gaps. Let x be an element of X^+. x is called an ω_α-limit of X^+ if ω_α is regular and α is the smallest ordinal for which x is the limit of an increasing sequence $\{x_\xi: \xi < \omega_\alpha\}$ of elements of X preceding x. Likewise, x is called an ω_β^*-limit of X^+ if ω_β is regular and β is the smallest ordinal for which x is the limit of a decreasing sequence $\{x_\xi: \xi < \omega_\beta\}$ of elements of X succeding x. x is a two-sided limit if it is both an ω_α- and an ω_β^*-limit for some α and β.

We now define the character of an element of X^+ in the following manner:

1. For every two-sided limit element $x \in X^+$, we say x has character $c_{\alpha\beta}$ if x is both an ω_α- and an ω_β^*-element.
2. If x is the first element of X^+ and x is an ω_β^*-limit, we assign it the character $c_{\beta\beta}$.
3. If x is the last element of X^+ and x is an ω_α-element, we assign it the character $c_{\alpha\alpha}$.

An element of X^+ has a symmetric character if it is of the form $c_{\alpha\alpha}$ for some α.

Example 3. We now proceed with the main construction of this paper. To produce a nearly pseudocompact space that is not pseudocompact, we must construct a linearly ordered space in which every element is a two-sided limit of character c_{00}, c_{11}, or c_{22}. At every stage of the construction we will create an element with one of these three characters out of every element from the preceding stage which fails to be a two-sided limit.

Let

$$\phi = \underbrace{\omega_0 + 1 + \omega_0^*}_{M} + \underbrace{\omega_1 + 1 + \omega_1^*}_{N} + \underbrace{\omega_2 + 1 + \omega_2^*}_{P}$$

$$A = \omega_0^* + \phi + \omega_0$$

$$B = \omega_1^* + \phi + \omega_1$$

Note that ϕ is gap free, and the only gaps of A and B are end-gaps. Let $\phi_0 = \phi$. Between every pair of consecutive elements x_α and $x_{\alpha+1}$ in sections M and N of ϕ_0, place a copy of A, call it $A_{0\alpha}$. In section P, if x_β and $x_{\beta+1}$ are a pair of consecutive elements with $\omega_1 \le x_\beta, x_{\beta+1} \le \omega_1^*$, then place a copy of B between them, say $B_{0\beta}$. Otherwise, between every pair of consecutive elements in P we place a copy of A. Let

$$B_0 = \bigcup B_{0\beta} \quad \text{and} \quad A_0 = \bigcup A_{0\alpha}$$

and set

$$\phi_1 = A_0 \cup B_0 \cup \phi_0$$

with the order implicit in our construction. (That is, if x and y are two consecutive elements between which a copy of, say A, has been placed, then for every element $a \in A$, $x < a < y$.) Note that every element of ϕ_1 which was originally a member of ϕ_0 now has character c_{00}, c_{11}, or c_{22}, and therefore no longer has an immediate predecessor or successor in ϕ_1.

Our next objective is to construct ϕ_2. Each of the copies of A and B which were used to create two-sided limits in ϕ_1 in turn contains elements which must be made into two-sided limits. For the copies of ϕ contained in the middle of these copies of A and B, the construction of ϕ_2 is identical to that of ϕ_1: the copy of ϕ is divided into three sections and additional copies of A and B are inserted as before. Between pairs of consecutive elements not belonging to the middle segment ϕ of A or B, we place new copies of A. If A_1 and B_1 are the respective collections of the copies of A and B freshly inserted into ϕ_1, then

$$\phi_2 = A_1 \cup B_1 \cup \phi_1$$

with, as in the case of ϕ_1, the order implicit in our construction.

Likewise we construct recursively

$$\phi_n = A_{n-1} \cup B_{n-1} \cup \phi_{n-1}$$

Note that each element of ϕ_n previously in ϕ_{n-1} is either c_{00}, c_{11}, or c_{22}. Indeed, if an element of ϕ_{n-1} is already symmetric in character, it is not affected by the construction since it has no immediate predecessor or successor, and thus its character is preserved in the next step; if an element of ϕ_{n-1} is not symmetric, then this is precisely what the construction in the n^{th} step remedies. Furthermore each ϕ_n is free of gaps since ϕ and the interiors of A and B are gap-free.

Finally let

$$X = \bigcup_{n \in \omega_0} \phi_n.$$

Note that each element of X is either c_{00}, c_{11}, or c_{22}. If x is an element of X, then x is contained in some ϕ_N. Either x is carried over from ϕ_{N-1}, in which case it is now, by construction, a symmetric element, or it will become a symmetric element in the set ϕ_{N+1}, and its character will be preserved in every successive stage of the construction.

As far as gaps are concerned, since X is the countable union of ϕ_n's, and since each of these is gap-free, any gaps of X have character c_{00}. Thus the set X^+, the union of X with all its gaps, is a dense linearly ordered set in which every element is of character c_{00}, c_{11}, or c_{22}. Furthermore, by construction, X^+ is dense with elements of each character.

X^+ is not quite the example sought. If all of the c_{00} and c_{22} elements of the set X^+ constructed above are deleted, the result is a linearly ordered set with only c_{11} elements, all of whose gaps are either c_{00} or c_{22}. This is the final example and will be referred to as $\mathcal{H}$. It remains to show that $\mathcal{H}$ has the topological properties we desire.

Recall that p is a P-point of X if and only if every G_δ-set containing p is a neighborhood of p, and that X is a P-space if and only if all of its points are P-points.

Proposition 2. *Let X be a linearly ordered space. A point $p \in X$ with character $c_{\alpha\beta}$ is a P-point if and only if both $\alpha \neq 0$ and $\beta \neq 0$* (Gillman and Henriksen, 1954).

Proof. Suppose p is a P-point of character $c_{\alpha\beta}$ and assume without loss of generality that $\alpha = 0$. Since p is an ω_0-limit, let $\langle x_n \rangle$ be a sequence converging to p from the left. For each x_n, let U_{x_n} be an open set containing p that misses x_n. Then $\bigcap_{n\in\omega} U_{x_n}$ is a G_δ-set containing p that misses every x_n. Since every neighborhood of p must contain infinitely many x_n, it can't be a neighborhood.

Conversely, let $\bigcap_{n\in\omega} U_n$ be a G_δ-set containing p. We may assume the U_n's are descending. For $n \in \omega$, let $x_n \in U_n - U_{n+1}$. Then $\langle x_n \rangle$ is a sequence, which by hypothesis must not converge to p (otherwise p would be an ω_0 or ω_0^*-limit). Thus there is a neighborhood G of p which misses every x_n. Since our sequence was arbitrary, G must be properly contained in every U_n. Therefore $\bigcap U_{n\in\omega}$ is a neighborhood of p.

Corollary 3. *A linearly ordered space X is a P-space if and only if no element of X has character $c_{\alpha 0}$ or $c_{0\beta}$.*

Lemma 4. *Every P-space is nearly realcompact* (Schommer, 1994).

NOTE 1. All the elements of our space $\mathcal{H}$ have character c_{11}. It follows that $\mathcal{H}$ is a *P*-space and is thus nearly realcompact.

The nearly pseudocompactness of $\mathcal{H}$ will now follow easily from established results, but additional definitions will be required. An increasing or decreasing sequence $\{x_\xi\}_{\xi<\omega\alpha}$ of elements of X is called a *Q-sequence* if for every limit ordinal $\lambda < \omega_\alpha$, the limit of the sequence $\{x_\xi\}_{\xi<\lambda}$ is a gap of X. A gap is called a left (right) *Q-gap* if it is the limit of an increasing (decreasing) Q-sequence, and a *Q-gap* if it is both a left and a right Q-gap. The gap is a non-measurable *Q-gap* if the cardinalites of the Q-sequences that identify it as a Q-gap are non-measurable.

Lemma 5. *Let X be a linearly ordered space. Then X is realcompact if and only if every gap of X is a non-measurable Q-gap* (Gillman and Henriksen, 1954).

Lemma 6. *X is nearly pseudocompact and nowhere locally compact if and only if every relatively realcompact open set is empty* (Schommer, 1993).

NOTE 2. It now follows that $\mathcal{H}$ is nearly pseudocompact. To see this, let U be any relatively realcompact open subset of $\mathcal{H}$. By construction, U contains a c_{22} gap, say w. Since no gap of X is either an ω_1- or an ω_1^*-limit, no transfinite sequence converging to w can be a Q-sequence. U, then, contains a non-Q-gap and so every closed set containing U cannot be realcompact by Lemma 5. Therefore every relatively realcompact open subset is empty, and $\mathcal{H}$ is nearly pseudocompact and nowhere locally compact by Lemma 6. Finally note that since $\mathcal{H}$ nearly realcompact, it cannot be pseudocompact (otherwise it would be compact as well).

TWO THEOREMS OF HAUSDORFF

We could have simply appealed to two theorems of Hausdorff (1908), as did Gillman and Henriksen (1954), to claim the existence of a linearly ordered set whose elements and gaps are of sufficient character for a space to be nearly pseudocompact but not pseudocompact. Of course a pure existence claim does not provide insight into what such a space might look like. Nonetheless the old theorems of Hausdorff (1908) are interesting, and we might do well to present them here.

DEFINITION 3. Let U be the collection of element characters of a linearly ordered set, and let V be its collection of gap characters. We say that two dense linearly ordered sets are members of the same species if they have the same sets U and V. A species is denoted as an ordered pair (U, V). Furthermore let $W = U \cup V$ be the union of all element and gap characters. All species with the same set W are said to belong to the same genus. We denote the genus after its gap-free representative, namely $(W, \emptyset)$.

EXAMPLE 4. Our proto-example X^+ has species $(c_{00}c_{11}c_{22}, \emptyset)$. $\mathcal{H}$ has species $(c_{11}, c_{00}c_{22})$. Both of these linearly ordered set belong to the same genus.

DEFINITION 4. A linearly ordered set X is called irreducible if every open interval of X has the species of X.

Theorem 7. *Let X be a dense irreducible linearly ordered set of species (W, ∅). Then there exist dense irreducible linearly ordered sets of every species in the genus (W, ∅)* (Hausdorff, 1908).

DEFINITION 5. A set of characters W is called complete if there exist ordinals κ and λ such that:

1. For every $\alpha < \kappa$, there exists a $\beta < \lambda$ such that $c_{\alpha\beta} \in W$;
2. For every $\beta < \lambda$, there exists an $\alpha < \kappa$ such that $c_{\alpha\beta} \in W$;
3. W contains at least one symmetric element $c_{\sigma\sigma}$ where $\sigma < \min(\kappa, \lambda)$.

Theorem 8. *If W is a complete character set, then there exists a dense irreducible linearly ordered set with species (W, ∅)* (Hausdorff, 1908).

NOTE 3. With $\kappa = \lambda = 3$, the character set $W = \{c_{00}c_{11}c_{22}\}$ can be seen to be complete, and so by Theorem 7, a dense linearly ordered set of species $(c_{00}c_{11}c_{22}, \emptyset)$ exists. It now follows from Theorem 8 that a dense irreducible linearly ordered set of species $(c_{11}, c_{00}c_{22})$ exists.

Hausdorff's proof of Theorem 8 is quite long, but has the advantage of being constructive. In fact, though Hausdorff did things a bit differently, his proof has motivated the construction of our example $\mathcal{H}$. It is clear then by Theorems 7 and 8, that there exist examples of nearly pseudocompact, non-pseudocompact spaces of every regular cardinality $\geq\omega_2$.

LITERATURE CITED

BLAIR, R., AND E. VAN DOUWEN. 1992. Nearly realcompact spaces. Top. Appl., 47(3):209–221.

GILLMAN, L., AND M. HENRIKSEN. 1954. Concerning rings of continuous functions. Trans. Amer. Math. Soc., 77:340–362.

GILLMAN, L., AND M. JERISON. 1960. Rings of continuous functions. Van Nostrand, Princeton, New Jersey.

HAUSDORFF, F. 1908. Grundzüge einer theorie der geordneten mengen. Math. Ann., 65:435–505.

HENRIKSEN, M., AND M. RAYBURN. 1980. On nearly pseudocompact spaces. Top. Appl., 11:161–172.

PORTER, J., AND R. WOODS. 1988. Extensions and absolutes of Hausdorff Spaces. Springer-Verlag, New York.

SCHOMMER, J. 1993. Relatively realcompact sets and nearly pseudocompact spaces. Comm. Math. Univ. Carolina, 34(2): 375–382.

SCHOMMER, J. 1994. Fast sets and nearly realcompact spaces. Houston J. Math., 20:161–174.

WILLARD, S. 1970. General topology. Addison Wesley, Reading, Massachusetts.