A UNIQUENESS RESULT CONCERNING PRONY'S METHOD FOR FITTING LINEAR COMBINATIONS OF EXPONENTIALS

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ABSTRACT

The application of Prony's method to the problem of determining a linear combination of n exponentials $f(x) = \Sigma_{i=1}^{n} A_i e^{\alpha_i x}$ (A_i, α_i real) such that $f(x_j) = y_j$ for $j = 0, 1, \ldots, 2n-1$, where x_j and y_j are given real numbers with the $2n$ x-values equally spaced and the $y_j > 0$, is considered. Accounts of Prony's method in the literature fail to consider rigorously the question of uniqueness of the solution. Uniqueness is established in this paper in the sense that if there does not exist a linear combination of fewer than n exponentials satisfying the condition, then there exists at most one combination of n exponentials that does so, namely the solution obtained by applying Prony's method.

INTRODUCTION

The problem of fitting an exponential function

$$f(x) = \sum_{i=1}^{n} A_i e^{\alpha_i x} \tag{1}$$

to a set of data points (x_j, y_j), where both the A_i and the α_i are unknown, is important in many applications of mathematics. If the x-values are equally spaced, Prony's method may be applied, and it is generally suggested that the values of the $2n$ unknowns A_i, α_i, $i = 1, 2, \ldots, n$, are determined by requiring that

$$f(x_j) = y_j \tag{2}$$

for $2n$ data points (x_j, y_j), $j = 0, 1, \ldots, 2n-1$. Whereas the values of the A_i are uniquely determined once the α_i are known or specified, it is not clear that the α_i are uniquely determined. In the next section uniqueness is established in the sense that if there does not exist a linear combination with fewer than n terms satisfying the condition, then there exists at most one solution having n terms, namely the one obtained using Prony's method.

There is no restriction in assuming that $x_j = j$ ($j = 0, 1, \ldots, 2n-1$), and for this special case, Prony's method may be described as follows. Given the set of data points

$$\{(0, y_0), (1, y_1), \ldots, (2n-1, y_{2n-1})\},$$

a function $f(x) = \Sigma_{i=1}^{n} A_i e^{\alpha_i x}$ is sought with the property that $f(j) = y_j$, $j = 0, 1, 2, \ldots, 2n-1$. Prony (1795) is credited with observing that each of the $e^{\alpha_i x}$ ($i = 1, 2, \ldots, n$) satisfies an n-th order homogeneous linear difference equation with constant coefficients whose characteristic roots are

$$\beta_i = e^{\alpha_i} \quad (i = 1, 2, \ldots, n). \tag{3}$$

Thus $f(x)$ also satisfies this difference equation. If the difference equation is

$$f(j) + c_1 f(j+1) + \cdots + c_n f(j+n) = 0, \tag{4}$$

then by setting $j = 0, 1, 2, \ldots, n-1$ successively, one obtains the linear system

$$\begin{array}{lllll} y_0 & + c_1 y_1 + c_2 y_2 & + \cdots + c_n y_n & = 0 \\ y_1 & + c_1 y_2 + c_2 y_3 & + \cdots + c_n y_{n+1} & = 0 \\ \vdots & & & \\ y_{n-1} & + c_1 y_n + c_2 y_{n+1} & + \cdots + c_n y_{2n-1} & = 0. \end{array} \tag{5}$$

This system can be solved uniquely for the coefficients c_k ($k = 1, 2, \ldots, n$) if the determinant

$$D = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_2 & y_3 & \cdots & y_{n+1} \\ \vdots & & & \\ y_n & y_{n+1} & \cdots & y_{2n-1} \end{vmatrix} \tag{6}$$

is not zero. Once the c_k are found, the values of the β_i defined by Eq. 3 can be determined by setting $f(j) = \beta^j$ in Eq. 4. This procedure leads to the "characteristic equation"

$$1 + c_1 \beta + c_2 \beta^2 + \cdots + c_n \beta^n = 0. \tag{7}$$

If this equation has n positive roots, $\beta_1, \beta_2, \ldots, \beta_n$, then the exponents $\alpha_1, \alpha_2, \ldots, \alpha_n$ are determined by the relation (3), so that $\alpha_i = \ln \beta_i$ ($i = 1, 2, \ldots, n$). After the α_i are found, the A_i in Eq. 1 are determined from the linear system

$$\begin{array}{llll} y_0 & = A_1 & + A_2 & + \cdots + A_n \\ y_1 & = A_1 e^{\alpha_1} & + A_2 e^{\alpha_2} & + \cdots + A_n e^{\alpha_n} \\ \vdots & & & \\ y_{n-1} & = A_1 e^{(n-1)\alpha_1} & + A_2 e^{(n-1)\alpha_2} & + \cdots + A_n e^{(n-1)\alpha_n} \end{array} \tag{8}$$

which is derived by setting $f(j) = y_j$ for $j = 0, 1, \ldots, n-1$. If the α_i are distinct, then the determinant

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{\alpha_1} & e^{\alpha_2} & \cdots & e^{\alpha_n} \\ \vdots & & & \\ e^{(n-1)\alpha_1} & e^{(n-1)\alpha_2} & \cdots & e^{(n-1)\alpha_n} \end{vmatrix} \tag{9}$$

is not zero, because it is the Vandermonde determinant of the numbers $e^{\alpha_1}, e^{\alpha_2}, \ldots, e^{\alpha_n}$. Therefore the A_i are uniquely determined if n distinct values of the α_i are known or preassigned.

An algorithmic summary of Prony's method for fitting the data points (j, y_j), $j = 0, 1, \ldots, 2n-1$, by a function (1) is provided by the following sequence of steps:

(i) Solve the system (5) for the c_k ($k = 1, 2, \ldots, n$), provided the determinant (6) is not zero.

(ii) Find the roots of the characteristic equation (7).

(iii) If Eq. 7 has n positive roots $\beta_1, \ldots, \beta_n$, calculate the exponents $\alpha_1, \ldots, \alpha_n$ using the formula $\alpha_i = \ln \beta_i$.

(iv) Solve the linear system (8) for the coefficients $A_1, \ldots, A_n$, provided the determinant (9) is not zero.

UNIQUENESS OF THE SOLUTION

Suppose that $f(x) = \Sigma_{i=1}^{n} A_i e^{\alpha_i x}$ satisfies the condition $f(j) = y_j$, $j = 0, 1, 2, \ldots, 2n-1$, and let us assume that no linear combination of fewer than n exponentials satisfies the condition. It will be shown that each of the numbers $\beta_i = e^{\alpha_i}$ ($i = 1, 2, \ldots, n$) satisfies Eq. 7, where the c_k are determined by the system given by Eq. 5.

Consider the determinant D of Eq. 6 and its factorization given in Fig. 1. By their definition, the β_i are all nonzero. Our

$$D = \begin{vmatrix} \Sigma A_i\beta_i & \Sigma A_i\beta_i^2 & \cdots & \Sigma A_i\beta_i^n \\ \Sigma A_i\beta_i^2 & \Sigma A_i\beta_i^3 & \cdots & \Sigma A_i\beta_i^{n+1} \\ \vdots & & & \\ \Sigma A_i\beta_i^n & \Sigma A_i\beta_i^{n+1} & \cdots & \Sigma A_i\beta_i^{2n-1} \end{vmatrix}$$

$$= \begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ A_1\beta_1 & A_2\beta_2 & \cdots & A_n\beta_n \\ \vdots & & & \\ A_1\beta_1^{n-1} & A_2\beta_1^{n-1} & \cdots & A_n\beta_n^{n-1} \end{vmatrix} \begin{vmatrix} \beta_1 & \beta_1^2 & \cdots & \beta_1^n \\ \beta_2 & \beta_2^2 & \cdots & \beta_2^n \\ \vdots & & & \\ \beta_n & \beta_n^2 & \cdots & \beta_n^n \end{vmatrix}$$

$$= (A_1 A_2 \cdots A_n)(\beta_1 \beta_2 \cdots \beta_n) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \vdots & & & \\ \beta_1^{n-1} & \beta_2^{n-1} & \cdots & \beta_n^{n-1} \end{vmatrix} \begin{vmatrix} 1 & \beta_1 & \cdots & \beta_1^{n-1} \\ 1 & \beta_2 & \cdots & \beta_2^{n-1} \\ \vdots & & & \\ 1 & \beta_n & \cdots & \beta_n^{n-1} \end{vmatrix}$$

FIG. 1

assumption that n is the minimum number of exponential terms required to fit the data implies that all of the A_i are nonzero. So the factorization in Fig. 1 shows that the determinant D is nonzero. Thus the c_k are uniquely determined by the system (5). It follows from Cramer's rule that

$$c_j = D_j/D \qquad (10)$$

where

$$D_j = - \begin{array}{c} \begin{array}{ccccc} & & \text{(col. } j\text{)} & & \end{array} \\ \begin{vmatrix} y_1 & \cdots & y_0 & \cdots & y_n \\ y_2 & \cdots & y_1 & \cdots & y_{n+1} \\ \vdots & & & & \\ y_n & \cdots & y_{n-1} & \cdots & y_{2n-1} \end{vmatrix} \end{array} \qquad (11)$$

$$D_j = - \begin{array}{c} \begin{array}{ccccc} & & \text{col. } j & & \end{array} \\ \begin{vmatrix} \Sigma A_i\beta_i & \cdots & \Sigma A_i & \cdots & \Sigma A_i\beta_i^n \\ \Sigma A_i\beta_i^2 & \cdots & \Sigma A_i\beta_i & \cdots & \Sigma A_i\beta_i^{n+1} \\ \vdots & & & & \\ \Sigma A_i\beta_i^n & \cdots & \Sigma A_i\beta_i^{n-1} & \cdots & \Sigma A_i\beta_i^{2n-1} \end{vmatrix} \end{array}$$

$$= - \begin{vmatrix} A_1 & A_2 & \cdots & A_n \\ A_1\beta_1 & A_2\beta_2 & \cdots & A_n\beta_n \\ \vdots & & & \\ A_1\beta_1^{n-1} & A_2\beta_2^{n-1} & \cdots & A_n\beta_n^{n-1} \end{vmatrix} \begin{array}{c} \begin{array}{cccccc} & & & \text{col. } j & & \end{array} \\ \begin{vmatrix} \beta_1 & \beta_1^2 & \cdots & 1 & \cdots & \beta_1^n \\ \beta_2 & \beta_2^2 & \cdots & 1 & \cdots & \beta_2^n \\ \vdots & & & & & \\ \beta_n & \beta_n^2 & \cdots & 1 & \cdots & \beta_n^n \end{vmatrix} \end{array}$$

$$= -(A_1 A_2 \cdots A_n) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \beta_1 & \beta_2 & \cdots & \beta_n \\ \vdots & & & \\ \beta_1^{n-1} & \beta_2^{n-1} & \cdots & \beta_n^{n-1} \end{vmatrix} \begin{array}{c} \begin{array}{cccccc} & & & \text{col. } j & & \end{array} \\ \begin{vmatrix} \beta_1 & \beta_1^2 & \cdots & 1 & \cdots & \beta_1^n \\ \beta_2 & \beta_2^2 & \cdots & 1 & \cdots & \beta_2^n \\ \vdots & & & & & \\ \beta_n & \beta_n^2 & \cdots & 1 & \cdots & \beta_n^n \end{vmatrix} \end{array}$$

FIG. 2

Figure 2 shows the derivation of another expression for D_j. It follows from Figs. 1 and 2 that

$$c_j = \frac{D_j}{D} = -\frac{\begin{array}{c} \qquad\qquad (\text{col. } j) \\ \begin{vmatrix} \beta_1 & \beta_1^2 & \cdots & 1 & \cdots & \beta_1^n \\ \beta_2 & \beta_2^2 & \cdots & 1 & \cdots & \beta_2^n \\ \vdots & & & & & \\ \beta_n & \beta_n^2 & \cdots & 1 & \cdots & \beta_n^n \end{vmatrix} \end{array}}{\begin{vmatrix} \beta_1 & \beta_1^2 & \cdots & \beta_1^n \\ \beta_2 & \beta_2^2 & \cdots & \beta_2^n \\ \vdots & & & \\ \beta_n & \beta_n^2 & \cdots & \beta_n^n \end{vmatrix}}. \tag{12}$$

It will now be shown that each of the β_i satisfies the characteristic equation

$$1 + c_1\beta + c_2\beta^2 + \cdots + c_n\beta^n = 0.$$

Implementation of Prony's Method

The result established in the previous section is a mathematically precise answer to the question concerning uniqueness of a linear combination of n exponentials that fits a given set of $2n$ data points. From a practical point of view, however, this result does not lessen the well known possibility of computational difficulties arising in the application of Prony's method. This problem is of particular significance in connection with the use of the method to try to discover the "correct" number of exponential terms needed to fit a given set of data points. Lanczos (1956) pointed out, for example, that the two functions

$$f(x) = 0.0951\,e^{-x} + 0.8607\,e^{-3x} + 1.557\,e^{-5x}$$

and

$$g(x) = 2.202\,e^{-4.45x} + 0.305\,e^{-1.58x}$$

have functional values which agree to the second decimal place for $x = 0.05k$, $k = 1, 2, \ldots, 24$. This observation points to the need, in many cases, for extreme accuracy in the data and in calculations if Prony's method is to be used to recover the "correct" exponential function which fits a given set of data points.

$$\begin{vmatrix} \beta_1 & \beta_1^2 & \cdots & \beta_1^n \\ \beta_2 & \beta_2^2 & \cdots & \beta_2^n \\ \vdots & & & \\ \beta_n & \beta_n^2 & \cdots & \beta_n^n \end{vmatrix} - \begin{vmatrix} 1 & \beta_1^2 & \beta_1^3 & \cdots & \beta_1^n \\ 1 & \beta_2^2 & \beta_2^3 & \cdots & \beta_2^n \\ \vdots & & & & \\ 1 & \beta_n^2 & \beta_n^3 & \cdots & \beta_n^n \end{vmatrix} \cdot \beta$$

$$- \begin{vmatrix} \beta_1 & 1 & \beta_1^3 & \cdots & \beta_1^n \\ \beta_2 & 1 & \beta_2^3 & \cdots & \beta_2^n \\ \vdots & & & & \\ \beta_n & 1 & \beta_n^3 & \cdots & \beta_n^n \end{vmatrix} \cdot \beta^2 - \cdots - \begin{vmatrix} \beta_1^2 & \beta_1^3 & \cdots & \beta_1^{n-1} & 1 \\ \beta_2^2 & \beta_2^3 & \cdots & \beta_2^{n-1} & 1 \\ \vdots & & & & \\ \beta_n^2 & \beta_n^3 & \cdots & \beta_n^{n-1} & 1 \end{vmatrix} \cdot \beta^n = 0$$

FIG. 3

When this equation is multiplied by the determinant

$$\begin{vmatrix} \beta_1 & \beta_1^2 & \cdots & \beta_1^n \\ \beta_2 & \beta_2^2 & \cdots & \beta_2^n \\ \vdots & & & \\ \beta_n & \beta_n^2 & \cdots & \beta_n^n \end{vmatrix},$$

it becomes (using Eq. 12) the equation displayed in Fig. 3. Inspection shows that the left side of this equation is the expansion of the $(n+1)$-th order determinant

$$\begin{vmatrix} 1 & \beta & \beta^2 & \cdots & \beta^n \\ 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^n \\ 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^n \\ \vdots & & & & \\ 1 & \beta_n & \beta_n^2 & \cdots & \beta_n^n \end{vmatrix},$$

which is zero for $\beta = \beta_i$ $(i = 1, 2, \ldots, n)$. Hence each of the β_i satisfies Eq. 7 as claimed.

In view of the uniqueness result established in this paper, a practical procedure for fitting an exponential function of the form (1) to a set of data points (x_j, y_j) is as follows. First try $n = 1$, i.e., fit a single exponential to any two of the points. If this function does not satisfactorily fit the remaining points, try $n = 2$, i.e., fit a two-term exponential to any four of the points. Continue to increase n until either a satisfactory fit is obtained, or Prony's method fails to produce a function of the desired form, either because the coefficient matrix in Eq. 5 is singular, or the characteristic equation (7) does not have all positive real roots. In the unlikely event that this procedure terminates with a function $f(x)$ of the form (1) (consisting of n exponential terms) which fits all of the data points exactly, then by the uniqueness result, $f(x)$ is the unique function of this form with $\leqq n$ terms that interpolates the data points.

Since this paper is concerned with the question of existence and uniqueness of a linear combination of exponentials (1) that fits the given data set, it omits consideration of alternative fits to the data involving terms of the form $e^{x\ln(-\beta_i)}\cos\pi x$ corresponding to negative roots β_i of the characteristic equation or similar terms corresponding to imaginary roots. The interested reader will find a discussion of these cases in Kelly (1967, pp. 80–81).