SOME PROPERTIES OF CERTAIN HYPER-SOLIDS
HERTA TAUSCHIO FREITAG AND ARTHUR H. FREITAG
Hollins College, Hollins College, Virginia

INTRODUCTION

We shall discuss some metric relationships which characterize certain hyper-solids of \( n \) dimensions and result in some well-known elementary formulas being recognized as special cases of much more generalized forms. For instance, the perimeter and area of an equilateral triangle, as well as the surface and volume of a regular tetrahedron are four of the specific cases which are obtained from a coverall relationship.

THE HYPER-CUBE:

The definition of this solid is obtained by following an iterative procedure. Start with a point, a 0-dimensional entity. Apply straight line motion to it over a distance of \( a \) units. Then its trace forms a line segment, a 1-dimensional magnitude. By moving this line segment out of its 1-dimensional space in a direction perpendicular to that space over \( a \) units a square, a 2-dimensional element, is obtained. Similar motion of the square creates a cube, a 3-dimensional manifold. Likewise with the cube, a tesseract (the 4-dimensional hyper-cube) will be formed. Continue this process indefinitely and it will lead to a set of hyper-cubes of \( n \) dimensions, where \( n \in \{0, 1, 2, \ldots \} \).

Relationships:

1. Number of boundary manifolds:

\[
N_{n,i} = 2^{n-1} \binom{n}{i}
\]  

where \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) and \( \binom{n}{0} = 1 \) whenever \( b = 0 \).

Since \( n \) means the dimension of the solid under consideration, and \( i \) represents the dimensionality of the boundary manifold in question, \( i \geq n \).

Proof:

To obtain relationship (1), note first that, owing to the definition of the \( n \)-dimensional hyper-cube, the recurrence formulas

\[(1) \quad \text{Multiply both sides of (a) by } 2^{n-1}, \text{ and rewrite the resulting equation as}
\]

\[
2^{n-1} \binom{n}{i} = 2 \cdot 2^{n-1-1} \binom{n-1}{i-1} + 2^{n-1-1-1} \binom{n-1}{i-1-1}.
\]

This, however, becomes interpreted as:

\[
M_{n,i} = 2M_{n-1,i} + M_{n-1,i-1}.
\]

\[
M_{n,0} = 2M_{n-1,0} = 2^n \quad \text{if } n > 0
\]

\[
M_{0,0} = 1
\]

hold. These formulas enable one to construct Table I quite rapidly.

But

\[
\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}; \quad (a),
\]

as may be seen from the definition of these symbols. Superimposing this relationship on formulas (0), relationship (1) results. (1)

Formula (1) may now be used for some specific values of \( n \) and \( i \) to obtain a whole array of otherwise unrelated facts: a line segment has two points as its extremities (\( M_{1,0} = 2 \)); a square has four vertices (\( M_{2,0} = 4 \)), and four edges (\( M_{2,1} = 4 \)), and a cube has eight vertices (\( M_{3,0} = 8 \)), twelve edges (\( M_{3,1} = 12 \)) and six faces (\( M_{3,2} = 6 \)).

2. Relationships among these manifolds:

An horizontal summation in Table I results in:

\[
\sum_{i=0}^{n} M_{n,i} = 3^n
\]

(2)

Proof:

Explicitly stated, relationship (2) shows that

\[
N_{n,0} + N_{n,1} + \ldots + N_{n,i} + \ldots + N_{n,n} = 3^n,
\]

or—according to (1)—

\[
2^n + 2^{n-1}n + \ldots + 2^{n-1} \binom{n}{i} + \ldots + 2 \binom{n}{n-1} + 1 = 3^n.
\]

However, since the left member of this statement represents the binomial expansion of \((2 + 1)^n\), the conclusion follows immediately.

3. Euler’s generalized relationship:

In 3-dimensional space, \( V \), the number of vertices, \( E \), the number of edges and \( F \), the number of faces of any solid are interconnected by the well-known Euler-formula

\[
V + F = E + 2
\]

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This law, which—in our terminology—reads
\[ M_3,0 + M_3,2 = M_3,3 + 2, \]
can be generalized, at least in the case of hyper-cubes. It becomes:
\[ \sum_{i=0}^{1} (-1)^{i} n_i,1 = 3. \]

Proof:
In detail, (3) states that
\[ X_n = X_0 + X_1 + \cdots + (-1)^{n} X_{n-1}. \]
Similar to the above proof for equation (2), this means that
\[ Z_0 = Z_0 + Z_1 + \cdots + (-1)^{Z_0} Z_{n-1} = Z_{n-1}, \]
where the left side represents the binomial expansion of \((2-1)^n\) in the proof is established.

4. Length of hyper-diagonals:
The relationship
\[ D_{n} = a\sqrt{n} \]
where \(D_{n}\) is the length of the \(n\)-dimensional (hyper-)

diagonal of an \(n\)-dimensional hyper-cube, is immediately obvious. It is stated here for completeness sake.

5. Hyper-volumes:
From the definition of (hyper-) volumes, it is clear that
\[ V_n = V_{n-1} + V_{n-1} = \frac{a^n}{n!} \]
With the help of relationship (1), this may also be expressed as
\[ V_{n-1} = 2^{n-1} \frac{n!}{(n-1)!} \]

Selecting some specific values for \(i\) and \(n\), all of the following become special cases of the above:
A line segment (of length \(a\)) has 2 vertices \((V_1 = 2)\).
A square (whose side is \(a\)) has 4 vertices \((V_2 = 4)\).
A cube (whose side is \(a\)) has 8 vertices \((V_3 = 8)\).
The total length of its edges is 12a \((V_3 = 12a)\).
Its surface area measures \(6a^2 \)
and its volume contains \(a^3\) cubic units \((V_3 = a^3)\).

6. Circum- and inscribed hyper-spheres:

\[ R = \frac{a\sqrt{n}}{2} \]
and
\[ r = \frac{a}{2} \]

The Regular Hyper-Tetrahedron:
Start again with a point, i.e. an element of a point set of dimensionality zero. Take this point out of its space over a distance of \(a\) units to produce a line segment, a 1-dimensional configuration. It is defined by the original and the new position of the point.
Locate the center of this line segment. At this point erect a line perpendicular to the line segment and such that its endpoint is a unit from the extremities of the given line segment. This forms an equilateral triangle, a 2-dimensional entity, defined again by the two extremities of the line segment and the endpoint of the perpendicular. Find the center of this triangle. At this point draw a perpendicular to the plane of the equilateral triangle and such that its endpoint has a distance of \(a\) units from the vertices of the given triangle. The 3-dimensional solid created in this manner is a regular tetrahedron, uniquely determined by its vertices. Then obtain the center of this tetrahedron, using it as the footpoint of a line perpendicular to the 3-dimensional space of the tetrahedron and of such length that its endpoint has a distance of \(a\) units from the vertices of the regular tetrahedron. So we have a regular simplex, a 4-dimensional hyper-solid, which corresponds to the tetrasector in the former sequence of hyper-cubes. This process is repeated indefinitely for the following members of the set of \(n\)-dimensional regular hyper-tetrahedrons.

Here, the recurrence formulas
\[ M_{n,k} = M_{n-1,k} + M_{n-1,k-1} \quad \text{where} \quad n > 0, k > 1 \]
\[ M_{n,0} = M_{n-1,0} + 1 = n + 1 \quad \text{where} \quad n > 0 \]
\[ M_{n,0} = 1 \]
which translate the definition, are used to set up Table II.

Relationships:
1. Number of boundary manifolds:
\[ M_{n,1} = \binom{n + 1}{1} + 1 \]

Proof:
Similar to formula (1), (8) is based on
\[ \binom{n+1}{i} = \binom{n}{i-1} + \binom{n-1}{i} \]

Now let \(n\) be an odd natural number, and for \(i = \frac{n+1}{2} + 1\),

This, however, causes the above expression, the left side of (3), to equal 1.

4. Hyper-heights:
Limit the discussion to \(n\)-dimensional hyper-heights of \(n\)-dimensional regular hyper-tetrahedrons, symbolizing them by \(b_n\). This does not affect the generalization, since

\[ b_{n+1} = \frac{\sqrt{n}}{n!} \]

On the basis of the definition of these hyper-solids, the following observations may be made:

\[ b_1 = 1 \]
\[ b_2 = \frac{1}{2} \]
\[ b_3 = \frac{1}{2} \sqrt{2} \]

Applying mathematical induction, we get
\[ b_n = \frac{1}{n!} \sqrt{\binom{n}{1} + \frac{\binom{n}{2}}{2}} \]
\[ b_{n+1} = \frac{1}{(n+1)!} \sqrt{\binom{n+1}{2}} \]

Proof:
This holds for all \(n \geq 2\) and is an immediate consequence of (10) and (1).
and thus relationship (10) is justified.

Furthermore, one formula establishes $b_3$, the height of an equilateral triangle as $\frac{a}{\sqrt{3}}$, and $b_4$, the (3-dimensional) height of a regular tetrahedron as $\frac{a}{\sqrt{6}}$.

5. Hyper-volumes:

First, we show that $V_{n,n}$, the n-dimensional content of our regular tetrahedron, is given by:

$$V_{n,n} = \frac{a^n}{\sqrt{6} n!}$$

(11)

Proof:

For the first equality of (11), an inductive procedure is used. It is evident that:

$$V_{1,1} = \frac{1}{2} a_1 b_1$$

$$V_{2,2} = \frac{1}{2} a_2 b_2 \sqrt{3}$$

$$V_{3,3} = \frac{1}{2} a_3 b_3 \sqrt{4}$$

hence,

$$V_{n,1} = \frac{a_1 b_1}{2^n \sqrt{1!}}$$

$$V_{n,n+1} = \frac{a b}{2^n \sqrt{2!}}$$

$$V_{n,2} = \frac{a b}{2^n \sqrt{3!}}$$

$$V_{n,n} = \frac{a b}{2^n \sqrt{n!}}$$

which equals $V_{n,n}$.

Furthermore, $V_{n,n}$, the (n-dimensional) hyper-volume of our n-dimensional tetrahedrons obeys the law that:

$$V_{n,1} = \frac{a_1 b_1}{2^n \sqrt{1!}}$$

$$V_{n,n+1} = \frac{a b}{2^n \sqrt{2!}}$$

$$V_{n,2} = \frac{a b}{2^n \sqrt{3!}}$$

and

$$V_{n,n} = \frac{a b}{2^n \sqrt{n!}}$$

Proof:

The first part of this continued equation utilizes the definition of hyper-contents. The remaining equalities follow by substitution from (11) and (8).

Once again, we realize the wealth of information contained in (12). It states—among many other things—that our original line segment has 2 vertices and length $c$ ($V_{1,1}=2$, $V_{1,1}=a$), that our equilateral triangle has 3 vertices, perimeter $3a$ and area $\frac{3a^2}{\sqrt{3}}$ ($V_{2,2}$ and $V_{2,2}$ respectively). Our regular tetrahedron has 4 vertices, or as the total length of its edges, a surface of $a^2 \sqrt{3}$ and a volume of $\frac{3a^3}{4}$ (by computing $V_{3,3}$ for $n=0, 1, 2, 3$).

6. Relationship between hyper-volumes and hyper-heights:

It may be interesting to note that

$$V_{n,n} = \frac{1}{n!} \int_{-1}^{1} \int_{-1}^{1} \ldots \int_{-1}^{1} d^n x$$

which combines some previously established results.

7. Circumscribed and inscribed hyper-spheres:

(a) Relationship between hyper-radii and hyper-heights:

$$R_n = \frac{a}{n!} \int_{-1}^{1} \int_{-1}^{1} \ldots \int_{-1}^{1} d^n x$$

$$r_n = \frac{a}{(n+1)!} \int_{-1}^{1} \int_{-1}^{1} \ldots \int_{-1}^{1} d^n x$$

where $R_n$ and $r_n$ symbolize the hyper-radius of the n-dimensional circumscribed and inscribed hyper-spheres, respectively. Thus,

$$R_n + r_n = h_n$$

(13a)

Proof:

(13) may be proved by mathematical induction, starting from the initial observation

$$h_2 = \frac{a}{2} \sqrt{2}$$

and

$$h_3 = \frac{3}{2} \sqrt{4}$$

In like manner, $R_n$ is found.

(b) Relationship between hyper-radii and sides:

$$R_n = \frac{a}{n!} \sqrt{2(n+1)}$$

$$r_n = \frac{a}{(n+1)!} \sqrt{2(n+1)}$$

(14)

and

$$\frac{R_n}{r_n} = n$$

(14a)

Proof:

(14) may be recognized quite readily by using some of the earlier relationships (13) and (10). (14) follows from combining both forms of (14).

This generalized relationship enables us to obtain the radii of the circumscribed and in-circles of our equilateral triangle.

$$R_2 = \frac{a \sqrt{3}}{\sqrt{2}}$$

and

$$r_2 = \frac{a \sqrt{3}}{\sqrt{6}}$$

as well as

the radii of circumscribed and inscribed spheres of our tetrahedron as

$$R_3 = \frac{a \sqrt{6}}{4}$$

and

$$r_3 = \frac{a \sqrt{6}}{12}$$

Corollary:

The ratio between the volumes of the two respective hyper-spheres is given by

$$\frac{\text{circumscribed hyper-sphere}}{\text{inscribed hyper-sphere}} = n$$

(14b)

Proof:

The reasoning is analogous to the proof for (6b). This points up a very rapid increase in this ratio for higher dimensions. Thus, the area of a circle circumscribed about an equilateral triangle is only 4 times as large as the area of its in-circle, the hyper-volume of an hyper-sphere circumscribed about a regular simplex is 256 times as large as the hyper-volume of the in-scribed hyper-sphere.