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HYPERGEODESICS WHICH HAVE NO CUSP-AXES[1]

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1. Introduction. The curves defined on a surface S by a differential equation of the form

$$v'' = A + Bv' + Cv'^2 + Dv'^3 \qquad \left(v' = \frac{dv}{du}, \dots \right), \tag{1.1}$$

in which the coefficients A,B,C,D are functions of u,v, are called hypergeodesics. The envelope of the osculating planes at a point P_x of all the hypergeodesics (1.1) that pass through P_x is a cone which is ordinarily of the third class. In this case the cone has three distinct cusp-planes which intersect in a line called the cusp-axis of the hypergeodesics at the point P_x We are concerned here with the situation in which the cone is of class two and has no cusp-axis.

Two families of hypergeodesics, called ρ- and σ- tangeodesics, which have no cusp-axes, have been introduced and studied by Bell [1; p.575][1]. These curves are found to be associated in an interesting manner with the edges of Green, the directrices of Wilczynski, and the projective normal of Fubini. It would seem to be of interest to pursue these ideas further to include the investigation of other families of hypergeodesics for which the envelope of the osculating planes of all the curves of the family that pass through a point is a cone of the second class. In this paper we introduce three examples of such families of hypergeodesics and give new geometric characterizations for the scroll directrices of Sullivan and certain classical canonical lines.

2. Analytic basis. Let the differential equations of an analytic non-ruled surface S in ordinary projective space be written in the Fubini canonical form [2; p.69]

$$\begin{aligned} x_{uu} &= px + \theta_u x_u + \beta x_v, \\ x_{vv} &= qx + \gamma x_u + \theta_v x_v \end{aligned} \qquad (\theta = \log \beta\gamma). \tag{2.1}$$

We select an ordinary point P_x of the surface S as one vertex of the usual local tetrahedron of reference. When a curve C_λ through the point P_x is regarded as being embedded in a one-parameter family of curves defined by the equation

$$dv - \lambda(u,v)du = 0, \tag{2.2}$$

the local equation of the osculating plane at the point P_x of the curve C_λ is

$$2\lambda(\lambda x_2 - x_3) + (\lambda' + \beta - \theta_u\lambda + \theta_v\lambda^2 - \gamma\lambda^3)x_4 = 0, \tag{2.3}$$

in which we have placed $\lambda' = \lambda_u + \lambda\lambda_v$.

[1]Numbers in brackets refer to the references cited at the end of the paper.

Two lines $l_1(a,b)$, $l_2(a,b)$ are reciprocal lines [2;p.82] at a point P_x of a surface if the line $l_1(a,b)$ joins the point P_x and the point y defined by placing

$$y = -ax_u - bx_v + x_{uv}$$

and the line $l_2(a,b)$ joins the points ρ, σ defined by

$$\rho = x_u - bx, \quad \sigma = x_v - ax,$$

where a,b are functions of u,v. As the point P_x varies over the surface, the lines $l_1(a,b)$, $l_2(a,b)$ generate two reciprocal congruences Γ_1, Γ_2, respectively.

The two reciprocal lines $l_1(a,b)$, $l_2(a,b)$ are canonical lines $l_1(k)$, $l_2(k)$ of the first and second kind respectively in case

$$a = -k\psi, \quad b = -k\phi,$$

where k is a constant and ϕ, ψ are defined by

$$\phi = (\log\beta\gamma^2)_u, \quad \psi = (\log\beta^2\gamma)_v.$$

Canonical lines of the first kind lie in the canonical plane whose local equation is

$$\phi x_2 - \psi x_3 = 0. \tag{2.4}$$

A curve C_λ of the family (2.2) is a hypergeodesic on the surface S if the function λ satisfies a differential equation of the form (1.1). The cusp-axis of the hypergeodesics is the line $l_1(a,b)$ joining the point P_x to the point (0,-a,-b,1), where

$$a = \tfrac{1}{2}(\theta_v + C), \quad b = \tfrac{1}{2}(\theta_u - B).$$

The class of the cone reduces to two if either $A+\beta$ or $D-\gamma$ is equal to zero (but not both), and in this case the cone has no cusp-axis. Moreover, the class of the cone reduces to one, so that the cone becomes a straight line if, and only if, $A+\beta = D-\gamma = 0$. In this case the hypergeodesics (1.1) are the union curves of a general congruence Γ_1.

3. Hypergeodesics which have no cusp-axes. We now suppose that the class of the cone is two so that the family (1.1) of hypergeodesics has no cusp-axis at the point P_x. With Bell [1;p.577], we call the family of hypergeodesics represented by the differential equation

$$\lambda' = A_1 + B_1\lambda + C_1\lambda^2 + D_1\lambda^3, \tag{3.1}$$

where $A_1 = -\beta$, $D_1 \neq \gamma$, a u-polar family of hypergeodesics. Similarly, the differential equation

$$\lambda' = A_2 + B_2\lambda + C_2\lambda^2 + D_2\lambda^3, \tag{3.2}$$

where $A_2 \neq -\beta$, $D_2 = \gamma$, may be regarded as representing a v-polar family of hypergeodesics.

By means of equation (2.3), together with equation (3.1), we find that the osculating plane at the point P_x of a hypergeodesic curve (3.1) has the local equation

$$2(\lambda x_2 - x_3) + [B_1 - \theta_u + 2(C_1 + \theta_v)\lambda + 3(D_1 - \gamma)\lambda^2]x_4 = 0. \tag{3.3}$$

The envelope of the osculating planes at P_x of all the curves of the u-polar family (3.1) is found from equation (3.3) to be the non-degenerate quadric cone whose equation is

$$[2x_2 + (\theta_v + C_1)x_4]^2 + 4(D_1 - \gamma)[2x_3 + (\theta_u - B_1)x_4]x_4 = 0. \tag{3.4}$$

In a similar manner, the envelope of the osculating planes of all the hypergeodesics of the v-polar family (3.2) through P_x is the quadric cone whose equation is

$$[2x_3 + (\theta_u - B_2)x_4]^2 - 4(A_2 + \beta)[2x_2 + (\theta_v + C_2)x_4]x_4 = 0. \tag{3.5}$$

The vertex of each of these cones is, of course, the point P_x. The cone (3.4) is tangent to the tangent plane at the point P_x of the surface S along the v-tangent at P_x, while the cone (3.5) has the u-tangent at P_x for its line of contact with the tangent plane of the surface.

The polar plane of the u-tangent with respect to the cone (3.4) intersects this cone, besides in the v-tangent at P_x, also in a generator which is called [1;p.578] the u-edge of the u-polar family (3.1) of hypergeodesics at P_x. The u-edge is the line $l_1(a,b)$ for which a and b are given by the formulas

$$a = \tfrac{1}{2}(\theta_v + C_1), \quad b = \tfrac{1}{2}(\theta_u - B_1). \tag{3.6}$$

Similarly, the polar plane of the v-tangent with respect to the cone (3.5) intersects this cone in the u-tangent and the v-edge of the v-polar family (3.2) at P_x. The v-edge is the line $l_1(a,b)$ for which

$$a = \tfrac{1}{2}(\theta_v + C_2), \quad b = \tfrac{1}{2}(\theta_u - B_2). \tag{3.7}$$

The tangent plane of the cone (3.4) along the u-edge intersects the tangent plane of the cone (3.5) along the v-edge in a line which is called [1;p.579] the joint-edge of the two families (3.1),(3.2) at P_x. The joint-edge is the line $l_1(a,b)$ for which

$$a = \tfrac{1}{2}(\theta_v + C_2), \quad b = \tfrac{1}{2}(\theta_u - B_1). \tag{3.8}$$

The polar plane of the u-tangent at P_x with respect to the cone (3.4) intersects the polar plane of the v-tangent at P_x with respect to the cone (3.5) in a line called [1; p.580] the polar-axis of the two families (3.1),(3.2) at P_x. The polar-axis is the line $l_1(a,b)$ for which

$$a = \tfrac{1}{2}(\theta_v + C_1), \quad b = \tfrac{1}{2}(\theta_u - B_2). \tag{3.9}$$

The plane determined by the u-edge and the v-edge of the two families (3.1), (3.2) intersects the canonical plane in a line $l_1(a,b)$ for which

$$a = K\psi, \quad b = K\varphi, \tag{3.10}$$

where K is defined by

$$K = \frac{1}{2}\,\frac{(\theta_v + C_1)(\theta_u - B_2) - (\theta_v + C_2)(\theta_u - B_1)}{\psi(B_1 - B_2) + \varphi(C_1 - C_2)}. \tag{3.11}$$

This line will be called the planar edge of the two families (3.1),(3.2) at P_x.

4. Applications. In this section three examples of u-and v-polar families of hypergeodesics will be adduced and studied. In the first place, the equations of the asymptotic osculating quadrics Q_u and Q_v [2;p.80] at a point P_x of a curve C_λ of the family (2.2) on the surface S are,respectively,

$$(4.1)\quad \begin{aligned} &2\lambda^3(x_2x_3 - x_1x_4) + 2\beta\lambda x_4(x_3 - \lambda x_2) + C x_4^2 = 0, \\ &2(x_2x_3 - x_1x_4) - 2\gamma\lambda x_4(x_3 - \lambda x_2) + D x_4^2 = 0, \end{aligned}$$

where we have placed

$$(4.2)\quad \begin{aligned} C &= \beta[\lambda' - \beta + (\varphi - \theta_u)\lambda - (2\psi - \theta_v)\lambda^2] - (\beta\gamma + \theta_{uv})\lambda^3, \\ D &= \gamma[-\lambda' - \gamma\lambda^3 + (\psi - \theta_v)\lambda^2 - (2\varphi - \theta_u)\lambda] - (\beta\gamma + \theta_{uv}). \end{aligned}$$

The quadric of Wilczynski, represented by the equation

$$(4.3)\quad x_2x_3 - x_1x_4 - \tfrac{1}{2}\theta_{uv}x_4^2 = 0,$$

and the asymptotic osculating quadric Q_v intersect, besides in the asymptotic tangents at the point P_x, also in a conic which lies in the plane whose equation is

$$(4.4)\quad 2\lambda(\lambda x_2 - x_3) + [-\lambda' + \beta - (\varphi - \theta_u)\lambda + (2\psi - \theta_v)\lambda^2 + \lambda^3\gamma]x_4 = 0.$$

This plane coincides with the osculating plane (2.3) at the point P_x of the curve C_λ if, and only if, the curve C_λ is a hypergeodesic of the u-polar family (3.1) for which

$$(4.5)\quad A_1 = -\beta,\ B_1 = \theta_u - \varphi,\ C_1 = \tfrac{1}{2}\psi - \theta_v,\ D_1 = 0.$$

If the asymptotic osculating quadric Q_u is used in place of Q_v, the plane containing the residual conic of intersection of the quadric (4.3) and Q_u coincides with the osculating plane of C_λ at P_x if, and only if, C_λ is a hypergeodesic of the v-polar family (3.2) for which

$$(4.6)\quad A_2 = 0,\quad B_2 = \theta_u - \tfrac{1}{2}\varphi,\quad C_2 = \psi - \theta_v,\quad D_2 = \gamma.$$

Application of the results of § 3 to the two families of hypergeodesics thus defined on S leads immediately to the following statement:

At a point on a surface, the u-edge of the u-polar family (3.1) of hyper - geodesics for which the coefficients are given by (4.5) and the v-edge of the v-polar family (3.2) for which the coefficients are given by (4.6) are the scroll directrices of Sullivan. The joint-edge of the two families is the first directrix of Wilczynski, the polar-axis is the first edge of Green, and the planar edge is the first canonical line with k = - 3/8.

Another example of u-and v-polar families of hypergeodesics may be adduced in the following way. Let us consider the quadric of Wilczynski (4.3) and the principal quadrics of Lane [3;p.705] represented by the equation

(4.7) $$x_2x_3 + x_4(-3x_1 - \tfrac{1}{2}\phi x_2 - \tfrac{1}{2}\psi x_3 + k_4x_4) = 0 \qquad (k_4 \text{ arbitrary }).$$

By eliminating x_1 from equations (4.3),(4.7) and setting the discriminant of the result equal to zero, we obtain

(4.8) $$k_4 = -\tfrac{1}{8}\phi\psi - \tfrac{3}{2}\theta_{uv}.$$

Thus the principal quadric for which the coefficient k_4 is given by (4.8) is characterized geometrically by the property that it intersects the quadric of Wilczynski in the asymptotic tangents through P_x and in a residual pair of straight lines instead of a residual non-singular conic.

The asymptotic osculating quadric Q_v for a curve C_λ intersects the principal quadric (4.7) for which k_4 is defined by (4.8) in two straight lines, besides the asymptotic tangents at P_x, if, and only if, C_λ is a curve of the u-polar family (3.1) of hypergeodesics for which the coefficients are given by

(4.9) $$A_1 = -\beta, \quad B_1 = \theta_u - \tfrac{3}{2}\phi, \quad C_1 = \tfrac{1}{2}\psi - \theta_v, \quad D_1 = 2\gamma.$$

When the asymptotic osculating quadric Q_u is used similarly in place of Q_v, C_λ is a curve of the v-polar family (3.2) of hypergeodesics for which the coefficients are given by

(4.10) $$A_2 = -2\beta, \quad B_2 = \theta_u - \tfrac{1}{2}\phi, \quad C_2 = \tfrac{3}{2}\psi - \theta_v, \quad D_2 = \gamma.$$

We now apply the results of §3 to obtain the following conclusion:

At a point on a surface, the u-edge of the u-polar family (3.1) of hypergeodesics for which the coefficients are given by (4.9) is the line $l_1(a,b)$ with

$$a = \tfrac{1}{4}\psi, \quad b = \tfrac{3}{4}\phi.$$

The v-edge of the v-polar family (3.2) for which the coefficients are given by (4.10) is the line $l_1(a,b)$ with

$$a = \tfrac{3}{4}\psi, \quad b = \tfrac{1}{4}\phi.$$

The joint-edge of the two families of hypergeodesics at P_x is the canonical line $l_1(k)$ for which $k = -3/4$, the polar-axis is the first edge of Green, and the planar edge is the first directrix of Wilczynski.

L. Green [4;p.559] has defined an axial quadric at a point P_x on a surface to be a quadric, not belonging to a Moutard pencil, which has second order contact with the surface at the point P_x for which the three osculating planes at P_x of the curve of intersection of the quadric with the surface are coaxial in a line l called the axis of the quadric. Moreover, he has shown [4;p.561] that the asymptotic osculating quadric Q_v at a point of a curve C_λ, nowhere tangent to a curve of Darboux, is an axial quadric if, and only if, λ is a solution of a differential equation of the form (3.1) in which the coefficients are defined by

(4.11) $A_1 = -\beta$, $B_1 = \theta_u - \frac{5}{6}\varphi$, $C_1 = \frac{1}{2}\psi - \theta_v$, $D_1 = -\gamma$.

Similarly, the asymptotic osculating quadric Q_u of C_λ in a non-Darboux direction at P_x is an axial quadric in case λ satisfies equation (3.2) for which the coefficients are given by

(4.12) $A_2 = \beta$, $B_2 = \theta_u - \frac{1}{2}\varphi$, $C_2 = \frac{5}{6}\psi - \theta_v$, $D_2 = \gamma$.

Green has written the equations of the quadric cones which are the envelopes of the osculating planes at P_x of the u-and v-polar hypergeodesics thus obtained. Moreover, he has shown that the polar-axis of the two families at P_x is the first edge of Green. To these results we may add the following statement:

<u>At a point on a surface, consider the</u> u-<u>and</u> v-<u>polar families</u> (3.1),(3.2) <u>of hypergeodesics for which the coefficients are defined respectively by the formulas</u> (4.11),(4.12). <u>The</u> u-<u>edge is the line</u> $l_1(a,b)$ <u>for which</u>

$$a = \frac{1}{4}\psi, \quad b = \frac{5}{12}\varphi,$$

<u>and the</u> v-<u>edge is the line</u> $l_1(a,b)$ <u>for which</u>

$$a = \frac{5}{12}\psi, \quad b = \frac{1}{4}\varphi.$$

<u>The joint-edge of the two families is the canonical line</u> $l_1(-5/12)$, <u>which has been called the first axis of Bompiani, and the planar edge is the first axis of Čech.</u>

References

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