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A NON-ASSOCIATIVE ALGEBRA GENERATED BY SQUARE MATRICES[1]

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INTRODUCTION

With the continued study of the theory of associative algebras, attention has been directed to non-associative algebras or to algebras where the associative law is replaced by a type of partial associativity. Many papers on this subject have appeared in the literature, probably the most notable among them being Cayley's (1881) algebra of order 8, an eight unit generalization of the real quaternions. L. E. Dickson (1930, pp. 16-18) was the first to obtain the properties of Cayley's algebra without computations. He also proved that right and left hand division, except by zero, is always possible and is unique, a fact which was overlooked by Cayley. An equivalent algebra had been discovered by J. J. Graves (1848) as early as 1844.

More recent contributions have been made by R. H. Bruck (1944) and N. Jacobson (1937). In a series of papers (1942a, 1942b) A. A. Albert obtains a generalization of many elementary properties of a non-associative algebra and where possible makes a correlation to the analogous properties of the associative algebras. In a still later paper (Albert, 1944) he has made use of the fact that if the multiplication of matrices is performed by multiplying row by row, column by column, or column by row, non-associative algebras are formed. A structure theory is developed for not only algebras involving matric multiplication but for any algebra involving an involution. Albert suggests that the algebras formed by non-associative matric multiplication may have some applications in problems occurring in the physics.

In this paper we shall consider a specific non-associative algebra suggested by a problem in finite dimensional vector spaces over the field of complex numbers. In consideration of such a vector space, we note that the vectors form an additive abelian group under vector addition. We define the inner product of two vectors α and β to be

$$(\alpha, \beta) = \sum_{i=1}^{n} a_i \bar{b}_i = AB^{CT} \qquad (1)$$

where the a_i and b_i are the coordinates of α and β with respect to a fixed basis for the vector space, and where A and B^{CT} are $(1 \times n)$ and $(n \times 1)$ matrices, respectively. The matrix B^{CT} is obtained from B by replacing each element of B by its complex conjugate and transposing the result. Since the a_i and b_i are numbers in a field F, (α, β) is a number in F; hence we have a mapping of the inner products (α, β) onto F. In case the vector space is of dimension one, this mapping becomes a mapping of $(V_1(F), V_1(F))$ onto F and hence onto $V_1(F)$. We may consider this mapping as an operation of multiplication in $V_1(F)$. Using the

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usual definition of addition in F together with this operation of multiplication, we have an algebra derived from F which is non-associative provided F is complex. However, we shall consider a somewhat more general algebra of this type wherein we replace F by the ring R of square matrices of order n where the matric elements are numbers in the complex field.

The algebra we shall presently discuss is among those covered by W. E. Jenner's (1950) definition of a non-associative ring, which he defines as an additive abelian group closed under a product operation with respect to which two distributive laws hold.

DEFINITIONS AND THEOREMS

We shall now consider an algebra $\mathfrak{A}$ generated by square matrices of order n with elements in the complex field.

DEFINITION 1. Let $\mathfrak{A}$ be a non-associative algebra consisting of elements A, B, C, ..., (these elements being $n \times n$ matrices over the complex field), closed with respect to two well-defined binary operations, addition (+) and multiplication (o), defined as follows:

1. Addition is defined as the usual addition for matrices.
2. Multiplication is defined by the relation

$$A \circ B = AB^{CT} \tag{2}$$

where AB^{CT} is the ordinary matric product of A and B^{CT}.

As a consequence of definition 1 we have the following theorem:

THEOREM 1. The set S of elements A, B, C, ... of $\mathfrak{A}$ satisfy the following statements:

1. S is an abelian group under the operation (+).

2a. S is closed under the operation (o).

b. With respect to the operation (o) there exists a right identity.

3. With respect to the operation (o) both right and left distributive laws hold.

We must show that the following statements are satisfied for arbitrary A, B, C, $\in \mathfrak{A}$:

I. $A + B$ is in $\mathfrak{A}$; (closure)

II. $A + (B + C) = (A + B) + C$; (associativity)

III. There exists an element $0 \in \mathfrak{A}$ such that $A + 0 = 0 + A = A$; (0 is defined to be the identity element with respect to the operation (+))

IV. $X + A = 0$ has a unique solution X; (inverse element)

V. $A + B = B + A$; (commutative law)

VI. $A \circ B$ is in $\mathfrak{A}$; (closure)

VIIr. $(B + C) \circ A = B \circ A + C \circ A$; (right distributive law)

VII . $A \circ (B + C) = A \circ B + A \circ C$; (left distributive law)

VIII. There exists a unique element $I \in \mathfrak{A}$ such that $A \circ I = A$. (right identity element)

Statements I through V follow as well-known properties of matrices based upon properties of the complex field which is the field of the elements of the matrix. Using the product operation defined by (2), statements VI through VIII are readily verified. Closure is evident, since by definition the product $A \circ B = C$ where C is a matrix or order n and hence an element of $\mathfrak{A}$.

VIIr. $(B + C) \circ A = (B + C)A^{CT} = BA^{CT} + CA^{CT} = B \circ A + C \circ A$

VII . $A \circ (B + C) = A(B + C)^{CT} = A(B^{CT} + C^{CT}) = AB^{CT} + AC^{CT} = A \circ B + A \circ C.$

We prove the following Lemma before verifying statement VIII.

LEMMA 1. If $X \circ A = 0$, and $A \circ Y = 0$ for an arbitrary $A \in \mathfrak{A}$, then $X = Y = 0$.

Assume $X \neq 0$, then there exists an element $x_{pq} \neq 0$ for some p, q. Since A is arbitrary, we may take $A = (a_{ij})$ where $a_{ij} = 0$ for $i \neq 1$, $j \neq q$: $a_{1q} = 1$. Then

$$(X \circ A)_{p1} = \sum_{j=1}^{n} x_{pj}\, \bar{a}_{1j} = x_{pq} \neq 0$$ ($\bar{a}_{1j}$ is the complex conjugate of a_{1j}). Hence we have at least one element in the product $X \circ A = 0$ which is not equal to zero, contrary to the hypothesis. Therefore we must conclude that X is the zero element. An argument similar to the one just employed will show that Y equals zero.

VIII. If I is the matrix whose principal diagonal elements are all 1's and whose non-diagonal elements are all zero then:

$$A \circ I = AI^{CT} = AI = A .$$

In order to show the uniqueness of I, assume there is a second identity J such that $A \circ J = A$.

Then $$A \circ I = A \circ J,$$
whence $$A \circ I - A \circ J = 0,$$
and $$A \circ (I - J) = 0.$$

Since A is arbitrary, we may apply Lemma 1, and we have

$$I - J = 0.$$

$$\therefore \quad I = J.$$

Thus we have verified statement VIII. We may remark that statements VIIr and VII are a direct consequence of the definition of the inner product in a finite dimensional vector space.

THEOREM 2. $\mathfrak{A}$ does not possess a left identity, and under the operation $(\circ)$ is non-associative and non-abelian.

The presence of a left identity would insure the existence of matrices A and B such that $A \circ B = B$. However, consideration of the following simple example will show the failure of a left identity. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are elements of the field F, and let

$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for any choice of elements a, b, c, d since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ can never equal } \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix}.$$

The following counter-example clearly shows the failure of associativity in $\mathfrak{A}$:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \circ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \circ \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \circ \begin{pmatrix} -1 & 0 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$$

The absence of a left identity assures the failure of the commutative law since $A \circ I = A$, while $I \circ A \neq A$, in general.

Albert (1942) has shown that the existence of a right and left identity is tantamount to asserting associativity. Consequently the absence of a left identity is assured since $\mathfrak{A}$ is non-associative and possesses a right identity. We may remark that $A \circ B = B$ has a solution $A = I$ if and only if B is hermitian, since

$$I \circ B = IB^{CT} = IB = B$$

and if $I \circ B = B$, we must have $B^{CT} = B$. It may happen that $A \circ B = B$ for some matrix not equal to I.

POLYNOMIALS WITH COEFFICIENTS IN $\mathfrak{A}$

We shall designate by $\mathfrak{A}[x]$ the set of all polynomials in x with coefficients in $\mathfrak{A}$. It is convenient to postulate that the indeterminate x be commutative with all elements of $\mathfrak{A}$, that is, $Ax = xA$ for each element $A \in \mathfrak{A}$. Furthermore, we shall assume that the commutative and associative laws hold for all indeterminates x.

DEFINITION 2. If

$$f(x) = A_n x^n + A_{n-1}x^{n-1} + \dots + A_o \tag{3}$$

and

$$g(x) = B_m x^m + B_{m-1}x^{m-1} + \dots + B_o \tag{4}$$

then their sum is a third polynomial such that if $m = n$,

$$S(x) = (A_n + B_n)x^n + (A_{n-1} + B_{n-1})x^{n-1} + \dots + (A_o + B_o). \tag{5}$$

If $m \neq n$, we may take $m > n$, then

$$S(x) = B_m x^m + \dots + B_{n+1}x^{n+1} + (A_n + B_n)x^n + \dots + (A_o + B_o). \tag{6}$$

We state the following definitions and theorems concerning the degrees of elements of $\mathfrak{A}[x]$.

DEFINITION 3. In the polynomial $f(x)$ the highest power of x which has a nonzero coefficient is called the degree of $f(x)$. (By this definition a nonzero element of $\mathfrak{A}$ is of degree zero in $\mathfrak{A}[x]$.) We shall assign to the zero element the degree minus infinity, in order that certain theorems concerning the degrees of polynomials shall hold without exception.

THEOREM 3. The degree of $f(x) + g(x)$ is not greater than the degree of $f(x)$ nor greater than the degree of $g(x)$.

Case 1. If $m = n$. By definition the sum of $f(x)$ and $g(x)$ can be represented by (5) which is a polynomial of degree not greater than m or n.

Case 2. If $m \neq n$. Suppose $m > n$. This sum can be represented by (6) which is a polynomial of degree not greater than m or n.

COROLLARY 1. If $m \neq n$, then the degree of $f(x) + g(x)$ is precisely the larger of the two degrees. This follows since the coefficient of the highest power of x in the polynomial of larger degree cannot be zero, and the coefficient of the term of highest degree in the sum is just the coefficient of the term of highest degree in the polynomial of larger degree. Hence the sum of the corresponding coefficients cannot be zero.

DEFINITION 5. If for a given nonzero element $A \in \mathfrak{A}$, there exists a nonzero element $B \in \mathfrak{A}$, for which $B \circ A = 0$, then A is defined to be a proper right divisor of zero.

DEFINITION 6. If for a given nonzero element $A \in \mathfrak{A}$, there exists a nonzero element $C \in \mathfrak{A}$, for which $A \circ C = 0$, then A is defined to be a proper left divisor of zero.

MacDuffee (1946) shows that a necessary and sufficient condition that the matrix A be a divisor of zero is that the determinant of A be zero. The elements of $\mathfrak{A}$ are $n \times n$ matrices, hence there must exist elements in $\mathfrak{A}$ which are proper divisors of zero. Since there are proper divisors of zero in $\mathfrak{A}$, there are certainly divisors of zero in $\mathfrak{A}[x]$.

THEOREM 4. The existence of proper right divisors of zero in $\mathfrak{A}$ implies the existence of proper left divisors of zero in $\mathfrak{A}$, and conversely.

By definition, if $B \circ A = 0$, where $A \neq 0$ and $B \neq 0$ are elements of $\mathfrak{A}$, then A is a right divisor of zero, similarly B is a left divisor of zero. The converse follows by the same argument.

DEFINITION 7. The product $f(x) \circ g(x)$ is defined to be a polynomial of the form:

$$f(x) \circ g(x) = \sum_{p=0}^{n} \sum_{q=0}^{m} (A_p \circ B_q) x^{p+q} . \tag{7}$$

THEOREM 5. The degree of $f(x) \circ g(x)$ cannot exceed the sum of the degrees of $f(x)$ and $g(x)$.

If we let $p = n - i$ and $q = m - j$ where $0 \leq i \leq n$ and $0 \leq j \leq m$ in (7) the product can be represented by

$$f(x) \circ g(x) = \sum_{j=0}^{m} \sum_{i=0}^{n} (A_{n-i} \circ B_{m-j}) x^{n+m-i-j} .$$

The power of x in any term of the product cannot exceed $n + m$ since i and j are both non-negative, therefore the degree of $f(x) \circ g(x)$ cannot exceed the sum of the degree of $f(x)$ and $g(x)$.

COROLLARY 2. If A_n and B_m are not both proper divisors of zero, the product polynomial has precisely one term of degree $n + m$.

This is the term of highest degree and is the degree of the polynomial.

COROLLARY 3. If A_n and B_m are proper divisors of zero the degree of the product polynomial can be less than $n + m$.

This follows since $A_n \circ B_m$ may equal zero.

THEOREM 1a. The set S_1 of elements $f(x), g(x), h(x), \ldots$ of $\mathfrak{A}[x]$ satisfy the following statements:

1. S_1 is an abelian group under the operation (+).

2a. S_1 is closed under the operation ($\circ$).

b. With respect to the operation ($\circ$) there exists a right identity.

3. With respect to the operation (o) both right and left distributive laws hold.

We must show that statements analogous to statements I through VIII, in the proof of theorem 1, hold for $\mathfrak{A}[x]$. Statements I through V follow from elementary properties of polynomials and from the fact that they hold for the elements of $\mathfrak{A}$. Statement VI follows from Definition 7. The product of two polynomials is a third polynomial with matric coefficients, hence we have closure.

VIIr. To prove that $\left[g(x) + h(x)\right] \circ f(x) = g(x) \circ f(x) + h(x) \circ f(x)$ we may write $f(x)$, $g(x)$ and $h(x)$ as follows:

$$f(x) = \sum_{i=0}^{n} A_i x^i \; ; \quad g(x) = \sum_{j=0}^{m} B_j x^j \; ; \quad h(x) = \sum_{k=0}^{r} C_k x^k \; .$$

Substituting in the left side of VIIr, we have

$$\left[\sum_{j=0}^{m} B_j x^j + \sum_{k=0}^{r} C_k x^k \right] \circ \sum_{i=0}^{n} A_i x^i$$

We can write the expression in brackets as

$$\sum_{=0}^{q} (\epsilon_\ell B_\ell + \eta_\ell C_\ell) x$$

where $\epsilon_\ell = 1$ if $\ell \leq m$; $\epsilon_\ell = 0$ if $\ell > m$

$\eta_\ell = 1$ if $\ell \leq r$; $\eta_\ell = 0$ if $\ell > r$

and q is the larger of the two integers m and r. Substituting in (8) we have

$$\left[\sum_{\ell=0}^{q} (\epsilon_\ell B_\ell + \eta_\ell C_\ell) x \right] \circ \sum A_i x^i$$

$$\sum_{i=0}^{n} \sum_{\ell=0}^{q} \left[(\epsilon_\ell B_\ell + \eta_\ell C_\ell) \circ A_i \right] x^{\ell+i}$$

$$\sum_{i=0}^{n} \sum_{\ell=0}^{q} (\epsilon_\ell B_\ell \circ A_i + \eta_\ell C_\ell \circ A_i) x^{\ell+i}$$

$$\sum_{\ell=0}^{m} \sum_{i=0}^{n} B_\ell \circ A_i x^{\ell+i} + \sum_{\ell=0}^{r} \sum_{i=0}^{n} C_\ell \circ A_i x^{\ell+i}$$

$$g(x) \circ f(x) + h(x) \circ f(x) \; .$$

Statement VIIℓ can be proved in the same fashion.

$$\sum_{i=0}^{n} A_i x^i \circ I = \sum_{i=0}^{n} (A_i \circ I) x^i$$

$$\sum_{i=0}^{n} A_i x^i \; .$$

The uniqueness of I in $\mathfrak{A}[x]$ follows immediately from the uniqueness of I in A.

THEOREM 2a. $\mathfrak{A}[x]$ does not possess a left identity, and under the operation (o) is non-associative and non abelian.

The lack of a left identity, associativity, and commutativity in $\mathfrak{A}$ assures their absence in $\mathfrak{A}[x]$, for by taking elements of degree zero the counter-examples given for $\mathfrak{A}$ apply.

directly to $\mathfrak{A}[x]$.

THEOREM 6. Euclidean algorithm (right) in $\mathfrak{A}[x]$. Let $f(x)$ and $g(x)$ be polynomials of degree n and m, respectively, where the coefficient of the highest power of $g(x)$ is not a proper divisor of zero, then there exist unique elements $q(x)$ and $r(x)$ such that

$$f(x) = q(x) \circ g(x) + r(x) , \qquad (9)$$

where the degree of $r(x)$ is less than m.

There are two cases to be considered.

Case 1. If $m > n$, then $q(x) = 0$ and $r(x) = f(x)$.

Case 2. If $m \leq n$, let us consider the polynomial $f_1(x)$ such that

$$f_1(x) = f(x) - \left[A_n \circ Cx^{n-m}\right] \circ g(x)$$

where the coefficient of x^n is $A_n - (A_n \circ C) \circ B_m = 0$. Since B_m is not a proper divisor of zero, there is an element C such that $(A_n \circ C) \circ B_m = A_n$ by virtue of the properties of matrices. If the degree of $f_1(x)$ is less than m, we may take $q(x) = A_n \circ Cx^{n-m}$ and $r(x) = f_1(x)$. If the degree of $f_1(x)$ is k m, and the coefficient of x^k in $f_1(x)$ is D, then

$$\begin{aligned} f_2(x) &= f_1(x) - (D \circ Cx^{k-m}) \circ g(x) \\ &= f(x) - \left[A_n \circ Cx^{n-m}\right] \circ g(x) - \left[D \circ Cx^{k-m}\right] \circ g(x) \\ &= f(x) - \left[A_n \circ Cx^{n-m} + D \circ Cx^{k-m}\right] \circ g(x) \ . \end{aligned}$$

The degree of $f_2(x)$ is less than k. If the degree of $f_2(x)$ is less than m, we may choose

$$q(x) = A_n \circ Cx^{n-m} + D \circ Cx^{k-m}, \ r(x) = f_2(x).$$

However, if the degree of $f_2(x)$ is greater than m, a finite number of repetitions of this process will yield a $q(x)$ and $r(x)$ which will satisfy equation (9). In order to show the uniqueness of the polynomials $q(x)$ and $r(x)$, suppose there exist a $q_1(x)$ and $r_1(x)$ such that

$$f(x) = q_1(x) \circ g(x) + r_1(x),$$

where the degree of $r_1(x)$ is less than m. Then

$$q(x) \circ g(x) + r(x) = q_1(x) \circ g(x) + r_1(x)$$

or

$$q(x) - q_1(x) \circ g(x) = r_1(x) - r(x) \ . \qquad (10)$$

The degree of $r_1(x) - r(x)$ is less than m, but by corollary 2 the degree of the left side of (10) can be less than m only if $q(x) - q_1(x)$. Hence, the representation in theorem 6 is unique.

The following example will illustrate the right Euclidean algorithm:

$$\text{Let } f(x) = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix} x^2 + \begin{pmatrix} -7 & 1 \\ 6 & 4 \end{pmatrix} x + \begin{pmatrix} 5 & 3 \\ 1 & 8 \end{pmatrix}$$

and

$$g(x) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} x - \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \quad \text{then}$$

$$q(x) = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad r(x) = \begin{pmatrix} 4 & 4 \\ 6 & 7 \end{pmatrix};$$

$$\begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix} x^2 + \begin{pmatrix} -7 & 1 \\ 6 & 4 \end{pmatrix} x + \begin{pmatrix} 5 & 3 \\ 1 & 8 \end{pmatrix} =$$

$$\left[\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}\right] \circ \left[\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} x - \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}\right] + \begin{pmatrix} 4 & 4 \\ 6 & 7 \end{pmatrix}.$$

The right Euclidean algorithm may or may not hold if the coefficient of $g(x)$ is a proper divisor of zero. The following example illustrates a case in which the algorithm fails to hold.

$$\text{Let } f(x) = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ and}$$

$$g(x) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We cannot find a $q(x)$ such that the product of the leading coefficients of $q(x)$ and $g(x)$ equals $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ since $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ is a proper divisor of zero and $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ is not a divisor of zero. If $f(x) = q(x) \circ g(x) + r(x)$ we shall call $r(x)$ the right remainder in the division of $f(x)$ by $g(x)$ in $\mathfrak{A}[x]$. If $r(x) = 0$ then $g(x)$ is said to be a right factor of $f(x)$ in $\mathfrak{A}[x]$.

THEOREM 7. Euclidean algorithm (left) in $\mathfrak{A}[x]$. Let $f(x)$ and $g(x)$ be polynomials of degrees n and m, respectively, where the coefficient of the highest power of $g(x)$ is not a proper divisor of zero, then there exist unique elements $p(x)$ and $s(x)$ such that

$$f(x) = g(x) \circ p(x) + s(x) \tag{11}$$

where the degree of $s(x)$ is less than m. The proof is similar to that of the previous theorem, the principal difference being that we associate the elements of $\mathfrak{A}$ from the right.

There are two cases to be considered.

Case 1. If $m > n$, then $p(x) = 0$ and $s(x) = f(x)$.

Case 2. If $m \leq n$, define the polynomial $f_1(x)$ such that

$$f_1(x) = f(x) - g(x) \circ \left[D \circ A_n x^{n-m}\right]$$

where the coefficient of x^n is $A_n - B_m \circ (D \circ A_n) = 0$. Since B_m is not a proper divisor of zero, there is an element D such that $B_m \circ (D \circ A_n) = A_n$. If the degree of $f_1(x)$ is k and $k < m$ we may take $p(x) = D \circ A_n x^{n-m}$ and $s(x) = f_1(x)$. If $k \geq m$, and the coefficient of x^k in $f_1(x)$ is E, then

$$\begin{aligned} f_2(x) &= f_1(x) - g(x) \circ \left[D \circ Ex^{n-m}\right]. \\ &= f(x) - g(x) \circ \left[D \circ A_n x^{n-m}\right] - g(x) \circ \left[D \circ Ex^{k-m}\right] \\ &= f(x) - g(x) \circ \left[D \circ A_n x^{n-m} + D \circ Ex^{k-m}\right] \end{aligned}$$

The degree of $f_2(x)$ is less than k. If the degree of $f_2(x)$ is less than m, we may choose

$$p(x) = D \circ A_n x^{n-m} + D \circ Ex^{k-m} \text{ and } r(x) = f_2(x).$$

However, if the degree of $f_2(x)$ is greater than m, a finite number of repetitions of this process will yield a $p(x)$ and $s(x)$ which will satisfy equation (11). The uniqueness of $p(x)$ and $s(x)$ can be proved by the method of theorem 6.

We shall call $s(x)$ the left remainder in the division of $f(x)$ by $g(x)$. If $s(x) = 0$, then $g(x)$ is said to be a left factor of $f(x)$.

SUMMARY

The algebra we have discussed in this paper is a particular algebra in which the elements are matrices of order n. In theorem 1, we have a set of statements which might well be considered as a set of postulates satisfied by the elements of $\mathfrak{A}$. This set, however, does not completely describe $\mathfrak{A}$ as is evidenced by the fact that theorems 7 and 8, while possible for $\mathfrak{A}$, will not follow from the statements of theorem 1 alone. This gap may be filled by adding the following postulate:

> Every element of $\mathfrak{A}$ is zero, a proper divisor of zero, or an element such that there exist solutions of equations $A \circ X = I$ and $Y \circ A = I$ where I is the right identity element.

The existence of divisors of zero in $\mathfrak{A}$ is dependent on the properties of the elements of $\mathfrak{A}$, and would not necessarily hold in non-associative algebras in general.

The algebra with which we have dealt satisfies the postulates set forth by Albert (1944) and Jenner (1950) and hence the theory developed in each of these papers would be applicable here. However, it is well to note that the algebra we have discussed is, in fact, more specialized than the algebras discussed in these papers, and therefore the results obtained here need not, in every case apply to all of the more general algebras of Albert and Jenner.

As defined, $\mathfrak{A}$ becomes associative if the elements of $\mathfrak{A}$ are diagonal matrices with elements from the real field. These elements form a subalgebra $\mathfrak{A}_1$, which is not only associative but commutative. A special case of the subalgebra $\mathfrak{A}_1$, is one in which the elements are of order one.

The development of $\mathfrak{A}[x]$ is analogous to the usual development of polynomial domains over fields. Again it is apparent that many of the properties of $\mathfrak{A}[x]$ do not necessarily hold for non-associative algebras in general. This is especially true since many of the properties of $\mathfrak{A}[x]$ are dependent on the elements of $\mathfrak{A}$. In our treatment of $\mathfrak{A}[x]$ we assumed that associativity holds for the indeterminate x, furthermore that x is commutative with all the elements of $\mathfrak{A}$. Hence, throughout our discussion we have purposely avoided treating the x as a variable or unknown quantity. The problem becomes much more complex if this is attempted since a linear polynomial of general form would be

$$A \circ x + x \circ B + C$$

and a quadratic would take the form

$$(A \circ x) \circ x + B \circ (x \circ x) + x \circ (C \circ x) + (x \circ D) \circ x + (x \circ x) \circ E + x \circ (x \circ F) + G \circ x + x \circ H + M.$$

The problem would still be difficult if unilateral polynomials such as

$$A \circ (x \circ x) + (B \circ x) \circ x + C \circ x + D$$

are considered, since it is apparent that the solutions of the unilateral matrix equation would not apply here due to lack of associativity in the terms of the equation of degree greater than one. This complexity throughout the entire paper is due to the non-commutativity and even more to the non-associativity of the elements of $\mathfrak{A}$.

The consideration of the polynomials over $\mathfrak{A}_1$, that is $\mathfrak{A}_1[x]$, becomes much easier, and the difficulty in using x as a variable or unknown quantity in $\mathfrak{A}_1$ vanishes. Consequently, in this case many theorems of polynomial domains over a field, such as the remainder theorem, would be valid.

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