Journal of the Tennessee Academy of Science
Volume 63, Number 1, January 1988

A SIMPLE PROOF THAT THE GROUP A_5 IS SIMPLE

John Kinloch
East Tennessee State University
Johnson City, Tennessee 37614

Abstract

A group is simple if it contains no proper normal subgroups. It is well known that the alternating group of even permutations, A_n, is simple if $n \geq 5$. In most beginning courses in modern algebra this result is either stated without proof or is proven with complicated conjugacy arguments plus an appeal to Sylow's Theorems which are seldom covered in a first course in algebra. In this paper we present a proof that A_5 is simple employing only simple counting techniques plus an elementary observation about subgroups of order 2.

Proof

A_5 has $5!/2 = 60$ elements. There are $5!/5 = 24$ circular permutations of 5 things taken 5 at a time. Thus A_5 has 24 5-cycles of the form $(abcde)$, each of order 5. There are $5 \cdot 4 \cdot 3/3 = 20$ circular permutations of 5 things taken 3 at a time. Thus A_5 has 20 3-cycles of the form (abc), each of order 3. A_5 has one identity element. Hence we have accounted for 45 of the 60 elements of A_5. There are $(5 \cdot 4/2)(3 \cdot 2/2)/2 = 15$ elements of the form $(ab)(cd)$ in A_5. Each such product of disjoint transpositions is easily seen to be of order 2. With these remaining 15 elements of A_5, which are elements of order 2 made up of products of disjoint transpositions, we have accounted for all 60 of the elements of A_5.

We state without proof the following well known and easy to prove result.

THEOREM. If N is a normal subgroup of index m in a group G, then g^m is in N for every element g in the group G.

Now suppose N is a normal subgroup of A_5. If N is a proper subgroup, then the index of N is 2, 3, 4, 5, 6, 10, 12, 15, or 30. We show none of these cases is possible. We dispose of the first 8 cases by a simple counting argument and treat the case of index 30 separately.

i. N is of index 2.
Then N consists of 30 elements. By the above theorem, the square of every element of A_5 is in N. Since $(abc) = (acb)^2$, the 20 3-cycles of A_5 are in N. The formula $(abcde) = (adbec)^2$ shows that the 24 5-cycles of A_5 are also in N. Thus N has at least 44 elements, which is false.

ii. N is of index 3.
Then N consists of 20 elements. Again, by the above theorem, the cube of every element of A_5 is in N. Since $(abcde) = (acebd)^3$, then the 24 5-cycles of A_5 are in N. Thus N contains at least 24 elements, which is false.

iii. The cases of index 4,5,6,10,12 and 15 are similarly disposed of by the following formulas:
Index 4. $(abcde) = (aedcb)^4$.
Index 5. $(abc) = (acb)^5$.
Index 6. $(abcde) = (abcde)^6$.
Index 10. $(abc) = (abc)^{10}$.
Index 12. $(abcde) = (adbce)^{12}$.
Index 15. $(ab)(cd) = [(ab)(cd)]^{15}$.

Thus N cannot be of index 2, 3, 4, 5, 6, 10, 12, or 15. The last remaining case is if N is of index 30. Then N is a subgroup of order 2.

We state without proof the following easy to prove observation.

LEMMA. A subgroup of order 2 is a normal subgroup in a group if and only if the subgroup is contained in the center of the group.

As noted earlier, the 15 elements of order 2 in A_5 are each expressible as a product of disjoint transpositions of the form $(ab)(cd)$. But in A_5, $\alpha\beta \neq \beta\alpha$ whenever α is of the form $(ab)(cd)$ and β is of the form (abc). Thus no element of order 2 is in the center of A_5 which provides the needed result.

COROLLARY. A_5 has no normal subgroups of index 30. Thus we have proven the result.

THEOREM. A_5 is simple.