MEASURING DIPOLE MOMENT OF A PERMANENT MAGNET AND THE EARTH'S LOCAL MAGNETIC FIELD

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ABSTRACT—As a means to determine the dipole moment of a permanent magnet, as well as the horizontal component of the Earth's local magnetic field, a cylindrical neodymium magnet with axial magnetic dipole moment is suspended from a thin vertical string with its axis horizontal. In the absence of other magnetic fields, it settles along the Earth's magnetic north-south line. A Helmholtz coil set was placed around the magnet, with its symmetry axis parallel to the Earth's field and with the magnet at its center. Rotating the magnet about an axis through the string results in a restoring torque, which when released, results in oscillations of the magnet with frequency that depends on the coil current. The frequency was measured for several coil currents. The resulting frequency versus current data is used to determine the dipole moment of the magnet. The horizontal component of the Earth's magnetic field is also determined from the same data. Measurement of the Earth's field thus obtained is in good agreement with values obtained using other methods. The dipole moment of the magnet was also measured using a second method that involves measurement of terminal speed of the magnet as it falls in a vertical cylindrical aluminum tube against a magnetic braking force of current induced in the tube.

The measurements described in this article provide a simple method to determine the dipole moment of a permanent magnet, as well as the horizontal component of the Earth's local magnetic field. The method used here employs classical mechanics principles taught in undergraduate Physics courses and modest instruments built by the student in the lab.

In this method an externally controlled uniform magnetic field of a Helmholtz coil is applied parallel to the horizontal component of the Earth's field. A cylindrical permanent magnet is suspended horizontally from a vertical string at the center of the Helmholtz coil and rotated by a small angle in the horizontal plane with respect to the direction of the Helmholtz field and then released. The magnet starts oscillating about an axis through the vertical string due to the restoring torque of the string and the combined magnetic fields of the Earth and the Helmholtz coil. The frequency of oscillation of the magnet is measured for various coil currents. The square of the frequency of oscillation is directly proportional to the coil current. This data is used to determine dipole moment of the magnet and also the horizontal component of the Earth's magnetic field.

The magnitude of the Earth's magnetic field shows variations from place to place over the surface of the Earth. Models exist that give the magnetic field components by latitude and longitude for various locations on Earth. The source of the Earth's magnetic field is believed to be a massive circulation of molten conducting matter deep inside the Earth powered by the dynamo effect (Lorrain, 1993; Roberts and Glatzmaier, 2000; Buffet, 2007). Although the magnitude of this field is very small, only about 0.5 Gauss, it is strong enough to divert the trajectories of very energetic charged particles coming from outer space to the polar regions.

A second method may also be used to measure the dipole moment of the magnet. The second method may also be used as an independent alternative of measuring the magnetic dipole moment. In the second method the magnet is dropped into a vertical cylindrical aluminum tube. As the magnet falls in the tube it generates currents in the tube whose magnetic force opposes its downward motion, and the magnet achieves a terminal speed soon after it inters the tube. When the magnet achieves terminal speed, the rate at which work is done by gravity on the magnet equals the rate of joule heat production by induced currents in the tube. Using this and treating the magnet as a point dipole, we can write its dipole moment in terms of the magnet's measured terminal speed and electrical properties and geometry of the tube as will be shown in second part of the next section. However, the second method may be used to measure the magnetic dipole moment without significant error only when the magnet is small and very strong.

THEORY

Method of Magnetic Torque—If a magnet with dipole moment $\vec{\mu}$ is placed in a uniform magnetic field $\vec{B}$ (Fig. 1), it experiences a magnetic torque given by:

$$\vec{\tau} = \vec{\mu} \times \vec{B} \tag{1}$$

Assume that the magnet is suspended from a thin vertical string of length L and torsion constant κ, with its symmetry axis horizontal and making an angle θ with the horizontal magnetic field B. Taking into account the torque of the magnet and that of the string, the rotational motion of the magnet in the horizontal plane is described by the equation:

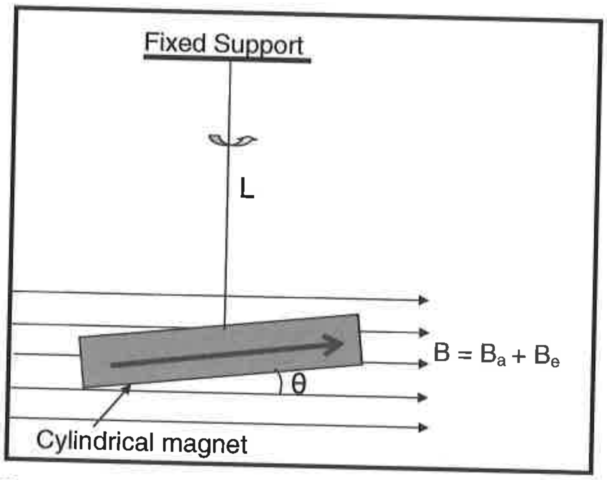

FIG. 1. Magnet is suspended from a vertical string and oscillates in a horizontal plane in a uniform and horizontal magnetic field.

$$\frac{d^2\theta}{dt^2} = -\frac{(\kappa/L)\,\theta + \mu B \sin\theta}{I} \qquad (2)$$

where

κ = Torsion constant of the string, L = Length of the string
$I = M(\ell^2/12 + R^2/4)$ = moment of inertia of magnet about axis through string.
M = mass of magnet, ℓ = length of magnet, and R = radius of the magnet

If the angle θ is small, equation (2) can be approximated by:

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa/L + \mu B}{I}\right)\theta \qquad (3)$$

This is identical in form to the Simple Harmonic Oscillator equation with the angular frequency ω given by:

$$\omega^2 = (2\pi f)^2 = \frac{\kappa/L + \mu B}{I} \qquad (4)$$

Writing the uniform magnetic field B as the sum of an externally controlled Helmholtz coil field B_a and the horizontal component of the Earth's magnetic field B_e, with the two fields parallel, we can write equation (4) as follows:

$$\omega^2 = (2\pi f)^2 = \frac{\kappa/L + \mu(B_a + B_e)}{I} \qquad (5)$$

We can relate the torque of the string to the torque of the Earth's magnetic field by measuring the frequencies of oscillation of the magnet at two different lengths of the string, with the Helmholtz field turned off.

If the magnet oscillates with period T_1 and T_2 at lengths L_1 and L_2, respectively, with the Helmholtz field turned off, equation (4) gives:

$$\omega_1{}^2 = (2\pi/T_1)^2 = (\kappa/L_1 + \mu B_e)/I \qquad (6)$$

and

$$\omega_2{}^2 = (2\pi/T_2)^2 = (\kappa/L_2 + \mu B_e)/I \qquad (7)$$

Combining equations (6) and (7) gives:

$$\kappa/L_2 = \left(\frac{\nu - 1}{\gamma - \nu}\right)\mu B_e = \eta\mu B_e \qquad (8)$$

where $\nu = (T_2/T_1)^2$, $\gamma = L_2/L_1$ and $\eta = (\nu - 1)/(\gamma - \nu)$

Equation (8) is used to relate the torque of the string to torque of the Earth's magnetic field at the length L_2 used during the experiment. Using equation (8) in equation (5), we get:

$$\omega^2 = (2\pi f)^2 = \frac{(1+\eta)\,\mu B_e + \mu B_a}{I} \qquad (9)$$

The Helmholtz field B_a at the center of the two identical coils carrying current in the same sense is given by (Reitz et al., 1992):

$$B_a = \frac{8}{\sqrt{125}}\frac{\mu_o N}{a} i \qquad (10)$$

where,

i = current in the coils
N = 130, the number of turns per coil,
a = coil separation = 0.148 m, and
$\mu_0 = 4\pi \times 10^{-7}\ T{\cdot}m/A$, is magnetic permeability of free space.

Using equation (10) in equation (9) gives:

$$f^2 = \frac{\mu}{4\pi^2 I}\left(\frac{8}{\sqrt{125}}\frac{\mu_0 N}{a}\right) i + \frac{\mu(\eta+1)}{4\pi^2 I} B_e \qquad (11)$$

which may be simplified to:

$$f^2 = C_1 i + C_2 = (\alpha\beta)\, i + \alpha(1+\eta)\, B_e \qquad (12)$$

where

$$\beta = \left(\frac{8}{\sqrt{125}}\frac{\mu_0 N}{a}\right)$$
$$C_1 = \alpha\beta, \text{ and } C_2 = \alpha(1+\eta)\, B_e \qquad (13)$$

Use of equations (12) and (13) gives the horizontal component of the Earth's local magnetic field and the dipole moment of the magnet in terms of the slope C_1 and the intercept C_2 of the graph of the square of frequency versus coil current as follows:

$$B_e = \left(\frac{\beta}{1+\eta}\right)\frac{C_2}{C_1} \qquad (14)$$

$$\mu = \frac{4\pi^2 I}{\beta} C_1 \qquad (15)$$

Method of Magnetic Braking—If a cylindrical magnet is dropped along its symmetry axis into a vertical cylindrical aluminum tube as shown (Fig. 3), induced current will circulate in the tube whose magnetic field opposes motion of the magnet by Lenz's law. The magnet attains terminal velocity

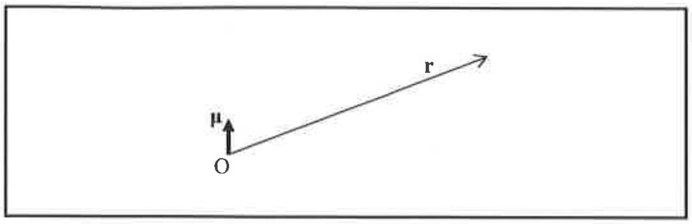

FIG. 2. Magnetic dipole at the origin.

v_t soon after it gets inside the tube. Once the magnet has attained terminal velocity, gravity does work on it at the following constant rate:

$$P_g = mg\, v_t \tag{16}$$

and this is converted to joule heat in the tube at the rate of:

$$P_e = \iiint \sigma E^2\, dV \tag{17}$$

where E is the electric field in the tube that is induced by the moving magnet and σ is conductivity of aluminum. The electric field is obtained from the magnetic vector potential $\vec{A}$:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial z}\frac{dz}{dt} = -v_z \frac{\partial \vec{A}}{\partial z} \tag{18}$$

The vector potential at $\vec{r}$ of a point magnetic dipole $\vec{\mu}$ that is placed at the origin (Fig. 2) with its direction parallel to the +z axis is given by (Rao, 1971; Karlsson, 1972)

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{\mu}\times\vec{r}}{r^3} \tag{19}$$

Equation (19) gives:

$$\frac{\partial A}{\partial z} = \frac{\partial}{\partial z}\left[\frac{\mu_0}{4\pi}\frac{\mu\rho}{(\rho^2+z^2)^{3/2}}\right] = \frac{3\,\mu_0\,\mu}{4\pi}\frac{\rho z}{(\rho^2+z^2)^{5/2}} \tag{20}$$

We will use equation (20) in equation (18) and integrate equation (17) using c and d for the limits on ρ, inner and outer radii, respectively, for the tube. We will take the limits of z to be $-\infty$ and $+\infty$. This is a good approximation since the magnetic field of the magnet is far from either end of the tube.

With these substitutions equation (17) reduces to:

$$P_e = (15/1024)\sigma\,(v_t\,\mu_0\,\mu)^2(1/c^3 - 1/d^3) \tag{21}$$

Equating equations (16) and (21) at terminal speed of the magnet, by conservation of energy, gives the magnetic dipole moment in terms of the conductivity σ and other quantities which can be measured directly.

$$mg = (15/1024)\sigma\; v_t\; (\mu_0\,\mu)^2\; (1/c^3 - 1/d^3) \tag{22}$$

where, m = mass of the magnet, σ = conductivity of aluminum, c = inner diameter of cylinder, d = outer diameter of cylinder, v_t = terminal speed of magnet, and $\mu_0 = 4\pi \times 10^{-7}$ Tm/A.

Using the measured numerical values of the quantities above in equation (22) gives the dipole moment of the magnet:

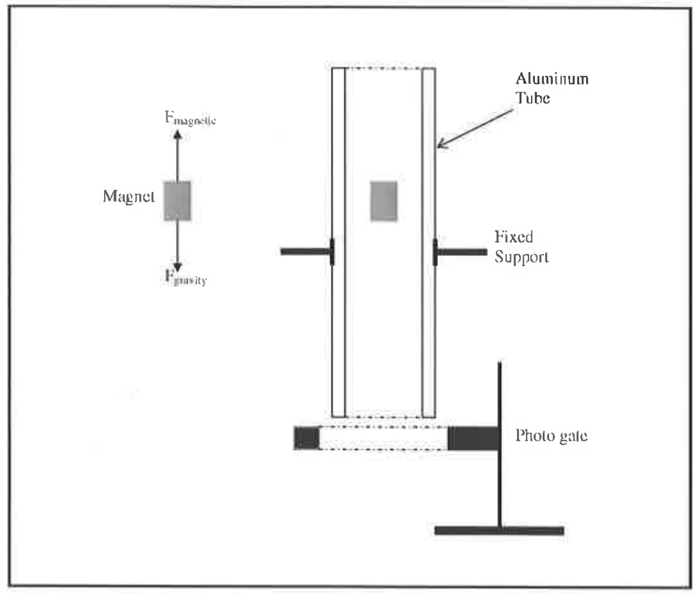

FIG. 3. The magnet falls at its terminal speed in a vertical aluminum cylinder.

EXPERIMENT

Method of Magnetic Torque

Equipment—Two Neodymium magnets (each with diameter of 1.2 cm, combined mass of 18.5 g, and combined length of 1.2 cm), a stop-watch timer, a thin string, a DC low voltage source, a set of Helmholtz coils of radius of 14.8 cm with 130 turns each, and a fluke digital multi-meter to measure coil current.

To measure contribution of the string to the torque on the magnet, the frequency of oscillation of the magnet was measured at two different lengths of the string with the Helmholtz field turned off. One of the lengths is L_1 = 43.5 cm while the second length L_2 = 87 cm which is the length of the string used for the experiment. The magnet was rotated in a horizontal plane by about 10° about a vertical axis through the supporting string relative to the Earth's magnetic field direction. The total times for 50 oscillations were measured ten times at each of these lengths. The average periods measured at L_1 and L_2 were T_1= 0.881 sec and T_2 = 0.953 sec, respectively.

These measurements and equation (8) are used to relate the string's contribution to the torque to that produced by the Earth's magnetic field:

$$\kappa/L_2 = \left(\frac{\nu-1}{\gamma-\nu}\right)\mu\, B_e = \eta\mu\, B_e = 0.203\mu\, B_e \tag{23}$$

where $\nu = T_2^2/T_1^2 = 1.169$ *and* $\gamma = L_2/L_1 = 2.0$

Equation (23) shows that the contribution of the string of length L_2 = 87 cm is about 20% of the torque of the Earth's magnetic field.

The frequency of oscillation of the magnet was also measured for various currents in the Helmholtz coil with the length of the string kept at L_2 = 87 cm. The magnet was suspended at the center of the Helmholtz coil with the symmetry axis of the coil aligned with the Earth's magnetic field. The total time for 24 cycles was measured manually using a stop watch and the period (time for one cycle) was

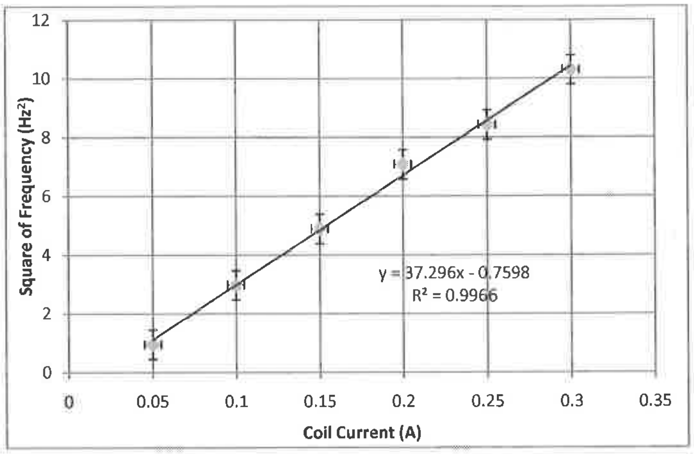

FIG. 4. Graph showing dependence of the square of frequency of magnet on current in the Helmholtz coil.

determined for various currents in the Helmholtz coil. The choice of 24 for total number of cycles is arbitrary.

The Helmholtz coil axis was aligned with the horizontal component of the Earth's magnetic field. But we found later that the Helmholtz field was set opposite to that of the Earth's magnetic field, because the current was set to flow opposite to the intended direction. Modifying equation (5), by changing the sign of $+B_e$ to take this into account and using equation (8) to substitute for the torque of the string, we obtain

$$f^2 = C_1 i + C_2 = (\alpha\beta) i + \alpha(\eta - 1) B_e \tag{24}$$

where

$$\alpha = \frac{\mu}{4\pi^2 I}, \ \beta = \left(\frac{8}{\sqrt{125}}\frac{\mu_0 N}{a}\right)$$

$$C_1 = \alpha\beta \quad \text{and} \quad C_2 = \alpha(\eta - 1) B_e \tag{25}$$

C_1 and C_2 are the slope and intercept, respectively, of the frequency squared versus coil current graph (Fig 4).

We can use equation (25) to write the dipole moment of the magnet and the horizontal component of the Earth's magnetic field in terms of the other measured quantities.

$$\mu = \frac{4\pi^2 I}{\beta} C_1$$

$$B_e = \left(\frac{\beta}{\eta - 1}\right)\frac{C_2}{C_1} \tag{26}$$

The values of C_1 and C_2 determined from a linear fitting of the graph have the following values:

$C_1 = 37.296$ Hz^2/A and $C_2 = -0.7598$ Hz^2

For the Helmholtz coil that is used for the experiment,

$$\beta = \left(\frac{8}{\sqrt{125}}\frac{\mu_0 N}{a}\right)$$

$= 0.000789817$ T/A, using n = 130 turns, and a = 14.8 cm

The moment of inertia of the magnet about an axis through the supporting string is:

$I = 3.306 \times 10^{-6}$ kg.m^2

Using these values in equation (26) gives the dipole moment of the magnet and the horizontal component of the Earth's magnetic field:

$$\mu = \frac{4\pi^2 I}{\beta} C_1 = (6.16 \pm 0.180)\ A.m^2$$

$$B_e = \left(\frac{\beta}{\eta - 1}\right)\frac{C_2}{C_1} = (0.202 \pm 0.057)\ Gauss$$

The horizontal component of the Earth's magnetic field reported for Martin, Tennessee, area is about 0.216 Gauss (National Geophysical Data Center). This is in fair agreement with our measurement.

The major sources of error in the magnetic torque method are in the measurement of the period of oscillation of the magnet and also in the measurement of the coil currents. The stop-watch measures time to within one-hundredth of a second, but the start and stop operation in the total timing could introduce even more error. The error in coil current is within about 5 milli-Amperes. Another source of error in using this method could be misalignment of the Helmholtz field and the Earth's horizontal field component. The magnetic torque method can be used with greater ease and with better precision with heavier and weaker magnets since such magnets oscillate more slowly and it allows a more accurate measurement of the period over a bigger range of Helmholtz coil currents.

Method of Magnetic Braking

Equipment—Two cylindrical Neodymium magnets with diameter of 1.30 cm and length of 0.60 cm, a stopwatch timer, Pasco photo gate timer, vertical tube support system, and a meter stick.

In this method the two magnets, joined along their length, are dropped into the top of the vertical aluminum cylinder. The cylinder has a length of 91.2 cm and inner and outer diameters of 1.87 and 2.54 cm, respectively. The terminal speed of the magnet was measured in two different ways. First, a stopwatch is used to measure the total time for the magnet to fall through the height of the cylinder. This was measured to be about 8.21 sec. Then a photo gate timer placed at the lower end of the cylinder as shown (Fig. 3) was used to measure the transit time of the magnet through the gate. Both methods gave the same value for the terminal speed: $v_t = 0.11$ m/s.

Equation (22) and the following measurements were used to evaluate the dipole moment of the magnet.

m = mass of the magnet = 13.5 g
σ = conductivity of aluminum
= 2.8×10^7 $(\Omega.\text{m})^{-1}$
c = inner diameter of cylinder = 2.0 cm
d = outer diameter of cylinder = 2.5 cm
v_t = terminal speed of magnet = 11.1 cm/s
$\mu_0 = 4\pi \times 10^{-7}$ Tm/A

Using the above numbers in equation (22) gives the magnetic dipole moment: $\mu = 4.50$ A.m^2 . This is about 73% of the value obtained using the magnetic torque method. Uncertainties in the value of conductivity of the cylinder and the terminal speed of the magnet contribute to the error in the dipole moment. But a major source of error in the magnetic braking method is the use of an oversimplified model. The vector potential given by equation (19) gives the magnetic field of a point dipole and it can be used to approximate the magnetic field of the given magnet only if the distance of the field point is much larger than the size of the magnet. Since the internal diameter of the cylinder is only about 1.5 times the diameter of the magnet, points in the tube may not be considered as far points in the calculation of the field. Thus, a more quantitative model that takes into account the near field contributions to the vector potential needs to be used. The other limiting factor of this method is that only very strong magnets with large enough dipole moments and small enough masses can reach their terminal speed within the length of the cylinder. We tried the magnetic braking method with a heavier and weaker magnet only to find that it had not reached its terminal speed when it came out of the lower end of the cylinder.

Conclusion

Our measurements show that the magnetic torque method can be used to measure magnetic dipole moments of many permanent magnets with very good accuracy. It can be used with magnets with arbitrary strength and size. While the major objective of the measurements is measuring magnetic dipole moment of a permanent magnet, the same data allow determination of the Earth's magnetic field. The method used here employs classical mechanics principles taught in undergraduate physics and modest instruments built by the student in the lab. In the magnetic torque method discussed in part A, the most important condition is that the applied magnetic field in the region surrounding the magnet be uniform. This condition is nearly met because Helmholtz coils produce almost uniform field over a region much larger than the size of the magnet around the center of the coils. The region occupied by the magnet has dimension less than one-tenth of the coil separation. Over a region within less than one-tenth of coil separation from the coil center, the axial magnetic field of a Helmholtz coil deviates from that at the center by less than one and half parts in ten thousand (Reitz et al., 1992). This method puts no restrictions on the strength of the magnet as long as it is small enough to lie in a reasonably uniform field region. This method was used with a weaker and larger cylindrical magnet and measurements of the frequency of oscillation were easier because it oscillates more slowly and it also allows use of higher coil currents. Thus the magnetic torque method can be employed to measure the magnetic moment of many permanent magnets and the local horizontal component of the Earth's magnetic field at the same time. This method also allows an easy way of determining the torsion constant of the string.

As discussed above, the major problem with the second method, the magnetic braking method, is use of an oversimplified model that treats the magnet as a point dipole for field calculations in the tube. The other problem with the method of magnetic braking is that only strong and light magnets can reach their terminal speed within the length of the tube.

Acknowledgments

We thank the editorial staff of the journal, including S. J. Stedman, C. T. Carter, and M. Weller, for suggesting very useful comments on the manuscript of this article. We also thank two anonymous reviewers for suggestions that improved the manuscript.

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Submitted 20 July 2009; accepted 30 November 2009.