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IMPROVEMENTS IN ADAMS-MOULTON METHODS FOR FIRST ORDER INITIAL VALUE PROBLEMS

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ABSTRACT—The conventional k-step Adams-Moulton formula is converted to a continuous form by using the integrand approximation process. For a specified step number k, k different finite difference formulas are recovered from the continuous form by evaluating it at some grid points. The k discrete formulas are simultaneously applied over successive blocks of meshes for a direct solution of nonstiff first order initial value problem. In this sense, the problem is solved without the need for any other methods to start the integration process.

In this paper, we develop a direct solution approach for solving the initial value problem of the form

$$y' = f(x, y), \qquad y(a) = y_0 \tag{1.1}$$

A solution is sought in the range $a \le x \le b$, where a and b are finite real numbers, and

$$a = x_0 < x_1 < x_2 \cdots < x_n = b$$

is a given discretization of [a, b]. Thus,

$$x_n = a + nh, \qquad n = 0, 1, 2, \ldots, N$$

where $N = (b - a)/h$ and h is a constant steplength.

We assume that f is Lipschitz continuous in y, that is that L exists so that

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2|, \qquad 0 < L < 1$$

for all $x \in [a, b]$ and for all y_1, y_2 in the region of interest. This condition ensures that (1.1) has a unique solution (Henrici, 1962). The general linear multistep method of step k is of the form

$$y_{n+k} = -\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1.2}$$

where α_j and β_j are constants and $|\alpha_0| + |\beta_0| \neq 0$. The main advantage of the linear multistep methods of the type (1.2) is that they are more economical than the one-step methods. For instance, they require fewer function evaluations of the derivative f than the one-step methods in the range of integration [a, b]. The conventional Adams Methods discussed in (Conte and De Boor, 1982; Gladwell and Sayers, 1980; Lambert, 1973) are special cases of (1.2) and are the most popular multistep methods used for solving (1.1). They are usually implemented as predictor-corrector methods using both an explicit and an implicit method to calculate y_{n+k}. In spite of their popularity, certain limitations of their applications are known as follows: They are discrete and therefore uneconomical for producing dense output. They are not self-starting and hence the starting values $y_1, y_2, \ldots, y_{k-1}$ are provided from other methods. This was done in the past by one-step methods like the Runge-Kutta method (Atkinson, 1989) and more recently by variable mesh/variable order methods. These are important limitations that can affect the performance of the Adams Methods.

The main objective of this paper is to present a block method for (1.1) which overcomes these limitations. In this light, we seek a continuous solution for (1.1). Recent research such as (Sarafyan, 1990; Jackson, 1988; Lie and Norset, 1989; Onumanyi et al, 1994) indicate growing interest in continuous integration algorithms for (1.1). Continuous Adams Methods can be derived by the method of collocation (Jator, 1992) and by the matrix inversion method (Onumanyi et al, 1994). However, higher order continuous Adams Methods are easily derived by the integrand approximation (Jackson, 1988; Jator, 1997) as in this paper. A Conventional k-step Adams-Moulton Method is converted to a continuous form and some finite difference formulas embedded in the latter are recovered. The finite difference formulas are then simultaneously applied as a block method for the direct solution of (1.1).

DERIVATION OF THE CONTINUOUS ADAMS-MOULTON METHODS

In this section, we construct the continuous forms of the Adams-Moulton Methods by the integrand approximation process. We consider the initial value problem (1.1) and the Newton's forward interpolation formula for a real analytic P(x) for these derivations, where P(x) is given by

$$P(x) = P(x_n + sh) = \sum_{j=0}^{\infty} [C_j^s \Delta^j f_n] \tag{1.3}$$

where $s = (x - x_n)/h$ and $C_j^s = (s!)/[(s-j)!j!]$.

In general, we let

$$t = \frac{(x - x_{n+k-1})}{h}$$

where k is the step number of the method. It can be shown that t is related to s as follows:

$$s = t + k - 1.$$

In constructing the Continuous Adams-Moulton Schemes, we integrated the differential equation (1.1) from $s = k - 1$ to

TABLE 1. Continuous Adams-Moulton Schemes for $1 \le k \le 5$.

K	j	βj(t)	βj(1)
1	0	$t - \frac{1}{2}t^2$	$\frac{1}{2}$
	1	$\frac{1}{2}t^2$	$\frac{1}{2}$
2	0	$-\frac{3}{12}t^2 + \frac{2}{12}t^3$	$-\frac{1}{12}$
	1	$\frac{12}{12}t - \frac{4}{12}t^3$	$\frac{8}{12}$
	2	$\frac{3}{12}t^2 + \frac{2}{12}t^3$	$\frac{5}{12}$
3	0	$\frac{2}{24}t^2 - \frac{1}{24}t^4$	$\frac{1}{24}$
	1	$-\frac{12}{24}t^2 + \frac{4}{24}t^3 + \frac{3}{24}t^4$	$-\frac{5}{24}$
	2	$\frac{24}{24}t + \frac{6}{24}t^2 - \frac{8}{24}t^3 - \frac{3}{24}t^4$	$\frac{19}{24}$
	3	$\frac{4}{24}t^2 + \frac{4}{24}t^3 + \frac{1}{24}t^4$	$\frac{9}{24}$
4	0	$-\frac{30}{720}t^2 - \frac{10}{720}t^3 + \frac{15}{720}t^4 + \frac{6}{720}t^5$	$-\frac{19}{720}$
	1	$\frac{180}{720}t^2 + \frac{40}{720}t^3 - \frac{90}{720}t^4 - \frac{24}{720}t^5$	$\frac{106}{720}$
	2	$-\frac{540}{720}t^2 + \frac{60}{720}t^3 + \frac{180}{720}t^4 + \frac{36}{720}t^5$	$-\frac{264}{720}$
	3	$\frac{720}{720}t + \frac{300}{720}t^2 - \frac{200}{720}t^3 - \frac{150}{720}t^4 - \frac{24}{720}t^5$	$\frac{646}{720}$
	4	$\frac{90}{720}t^2 + \frac{110}{720}t^3 + \frac{45}{720}t^4 + \frac{6}{720}t^5$	$\frac{251}{720}$
5	0	$\frac{36}{1440}t^2 + \frac{20}{1440}t^3 - \frac{15}{1440}t^4 - \frac{12}{1440}t^5 - \frac{2}{1440}t^6$	$\frac{27}{1440}$
	1	$-\frac{240}{1440}t^2 - \frac{120}{1440}t^3 + \frac{105}{1440}t^4 + \frac{72}{1440}t^5 + \frac{10}{1440}t^6$	$-\frac{173}{1440}$
	2	$\frac{720}{1440}t^2 + \frac{280}{1440}t^3 - \frac{330}{1440}t^4 - \frac{168}{1440}t^5 - \frac{20}{1440}t^6$	$\frac{482}{1440}$
	3	$-\frac{1440}{1440}t^2 - \frac{80}{1440}t^3 + \frac{510}{1440}t^4 + \frac{192}{1440}t^5 + \frac{20}{1440}t^6$	$-\frac{798}{1440}$
	4	$\frac{1440}{1440}t + \frac{780}{1440}t^2 - \frac{300}{1440}t^3 - \frac{375}{1440}t^4 - \frac{108}{1440}t^5 - \frac{10}{1440}t^6$	$\frac{1427}{1440}$
	5	$\frac{144}{1440}t^2 + \frac{200}{1440}t^3 + \frac{105}{1440}t^4 + \frac{24}{1440}t^5 + \frac{2}{1440}t^6$	$\frac{475}{1440}$

$s = t + k - 1$. We then replace the integrand $f(x, y)$ by the interpolating polynomial (1.3). The schemes are generated after truncating (1.3) to obtain polynomials of differing degrees and integrating. All the members of this class can compactly be expressed in the form

$$\bar{y}(t) = \bar{y}_{n+k-1} + h \sum_{j=0}^{k} \beta_j(t) f_{n+j} \qquad (1.4)$$

where $t = [(x - x_n)/h] - k + 1$, $t \neq 0$, $x_n \le x \le x_{n+k}$, $\bar{y}(t)$ is the continuous form and $\beta_j(t)$ are provided for $1 \le k \le 5$ (Table 1).

THE BLOCK ADAMS-MOULTON METHODS

We can obtain enough finite difference (FD) equations from the evaluation of $\bar{y}$ at points different from the interpolation points used in the construction of $\bar{y}$ for the integration of a nonstiff problem of the form (1.1). We evaluate (1.4) at the points

$$t = -(k - 1), -(k - 2), \ldots, -1, 1$$

to obtain k multiple discrete formulas which are all consistent and of close accuracies. We do not evaluate $\bar{y}$ given by (1.4) at $x = x_{n+k-1}$ ($t = 0$), because it is the only interpolation point used

TABLE 2. Absolute Errors of Methods for Example 1.

x	y_1-component		y_2-component	
	Standard AMM[1]	Block CAMM[2]	Standard AMM	Block CAMM
0.00	0.00	000	0.00	0.00
0.10	0.00[3]	4.87×10^{-6}	0.00	4.42×10^{-5}
0.20	4.87×10^{-6}	0.54×10^{-6}	4.44×10^{-5}	0.48×10^{-5}
0.30	10.80×10^{-6}	6.55×10^{-6}	9.90×10^{-5}	6.05×10^{-5}
0.40	17.40×10^{-6}	1.33×10^{-6}	16.50×10^{-5}	1.19×10^{-5}
0.50	25.90×10^{-6}	8.73×10^{-6}	24.50×10^{-5}	8.20×10^{-5}
0.60	35.90×10^{-6}	2.43×10^{-6}	34.00×10^{-5}	2.20×10^{-5}
0.70	47.80×10^{-6}	11.60×10^{-6}	45.40×10^{-5}	11.00×10^{-5}
0.80	61.70×10^{-6}	3.97×10^{-6}	58.90×10^{-5}	3.63×10^{-5}
0.90	78.10×10^{-6}	15.20×10^{-6}	74.80×10^{-5}	14.7×10^{-5}
1.00	97.30×10^{-6}	6.05×10^{-6}	93.40×10^{-5}	4.15×10^{-5}

[1] Adams-Moulton Method
[2] Continuous Adams-Moulton Method
[3] The theoretical solution is used as the starting value for y_1.

in (1.4). From (1.4), we obtain as an approximation to y_{n+i}, $i = 0, 1, \ldots, k$

$$\bar{y}_{n+i} = \bar{y}(t = i - (k - 1)), \quad i = 0, 1, \ldots, k - 2, \quad k \geq 2 \tag{1.5}$$

$$\bar{y}_{n+k} = y(t = 1) \tag{1.6}$$

As an illustration, we list the members of (1.5) and (1.6) for the case $k = 2$. From the continuous formula (1.4), we get

$$\bar{y}(t) = \bar{y}_{n+1} + \frac{h}{12}[(2t^3 - 3t^2)f_n + (-4t^3 + 12t)f_{n+1} + (2t^3 + 3t^2)f_{n+2}] \tag{1.7}$$

If $t = 1, -1$ we obtain respectively from (1.7)

$$\bar{y}_{n+2} \equiv \bar{y}(t = 1) = \bar{y}_{n+1} + \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n] \tag{1.8}$$

$$\bar{y}_n \equiv \bar{y}(t = 1) = \bar{y}_{n+1} + \frac{h}{12}[5f_{n+2} + 8f_{n+1} - f_n] \tag{1.9}$$

where (1.8) and (1.9) are solved simultaneously using the single block matrix equation to yield the results given as numerical examples.

THE SINGLE BLOCK MATRIX EQUATION

We can assemble all contributions from each block into a single larger block matrix equation for the solution of (1.1). To this end, we let $\tau = [N/k]$, ($\tau > 0$ denote the positive integer part for N/k and write

$$N = \tau k + r, \quad 1 \leq r < k.$$

The multiple finite difference formulas are simultaneously applied over the first τ successive blocks $\pi_0 \ldots, \pi_{\tau-1}$ of the form

$$\pi_l : \{[x_{kl}, x_{kl+1}], \ldots, [x_{kl+k-1}, x_{kl+k}]\}, \quad l = 0, \ldots, \tau - 1.$$

In each block π_l, we seek a discrete solution $\{\bar{y}_{kl+1}, \ldots, \bar{y}_{kl+k}\}$ simultaneously over successive steps where $\bar{y}_{kl}$ is known, presumably from the previous block, with $y(a) = y_0$. Then, the $\tau + 1$)st block which is the last block, is given by $\pi_\tau := \{[x_\ell, x_{\tau l+1}], \ldots, [x_{N-1}, x_N]\}$, where π_τ involves r successive steps. Varying the values of k in the continuous form formula $\bar{y}$ where π_τ involves r successive steps. Varying the values of k in the continuous form formula $\bar{y}$ for a constant value of h on the same mesh, leads to variable order multistep finite difference methods. The additional effort involved in the use of higher order methods is with the increased number of function evaluations in the matrix to be solved. For $k = 1, 2, \ldots$ the size of the matrix to be solved remains the same. The process provides $\bar{y}_1, \ldots, \bar{y}_{k-1}$ as well as $\bar{y}_k$ and so it is a self-starting method. For linear problems, we can solve the matrix equation directly from the start with Gaussian elimination using partial pivoting, while for non-linear problems, we use a modified Newton-Raphson iteration. This can be written in the form

$$J_v(\bar{y}^v - \bar{y}^{v-1}) = -F(\bar{y}^{v-1})$$

where F has N components, J_v is the Jacobian of F evaluated at $\bar{y}^{v-1}$ and v is an integer.

In all cases the set of linear equations or Jacobian has an almost block diagonal form.

NUMERICAL EXAMPLES

Example 1. (Nonstiff initial value problem)

$$y_1' = y_1, \quad y_1(0) = 1, \quad 0 \leq x \leq 1$$

$$y_2' = y_2 + e^x, \quad y_2(0) = 5$$

The theoretical solutions $y_1(x) = e^x$ and $y_2(x) = (5 + x)e^x$ are given here to compare with the accuracy of the numerical solutions in (Table 2).

Here the standard Adams-Moulton Method of order three is compared with the block Continuous Adams Moulton Method of order three. From Table 2 the use of the Continuous Adams-Moulton Method in block form should be preferable to the stan-

TABLE 3. Absolute errors of Continuous Adams-Moulton Method for order 2 and 3 for y_1 and y_2 components, $h = 0.1$, and $N = 10$.

x	y_1-component		y_2-component	
	CAMM[1] of order 2	CAMM of order 3	CAMM of order 2	CAMM of order 3
0.00	0.00	0.00	0.00	0.00
0.10	0.82×10^{-4}	0.64×10^{-6}	0.08×10^{-4}	4.11×10^{-6}
0.20	1.63×10^{-4}	0.44×10^{-6}	0.33×10^{-4}	0.09×10^{-6}
0.30	2.38×10^{-4}	1.86×10^{-6}	0.74×10^{-4}	3.77×10^{-6}
0.40	3.07×10^{-4}	8.17×10^{-6}	1.30×10^{-4}	0.35×10^{-6}
0.50	3.65×10^{-4}	2.96×10^{-6}	1.99×10^{-4}	3.11×10^{-6}
0.60	4.12×10^{-4}	1.10×10^{-6}	2.82×10^{-4}	0.75×10^{-6}
0.70	4.46×10^{-4}	3.86×10^{-6}	3.75×10^{-4}	2.18×10^{-6}
0.80	4.64×10^{-4}	1.24×10^{-6}	4.77×10^{-4}	1.27×10^{-6}
0.90	4.66×10^{-4}	4.49×10^{-6}	5.86×10^{-4}	1.02×10^{-6}
1.00	4.50×10^{-4}	1.20×10^{-6}	7.00×10^{-4}	1.87×10^{-6}

[1] Continuous Adams-Moulton Method.

dard Adams-Moulton Method in the step-by-step form for more accurate solution of nonstiff initial value problems.

Example 2. An initial value problem to demonstrate the self-starting nature of the Continuous Adams-Moulton Method.

$$y_1' = y_2, \quad y_1(0) = 0$$

$$y_2' = -y_1, \quad y_2(0) = 1, \quad x \in [0, 1],$$

where the theoretical solutions are given by $y_1(x) = \sin x$ and $y_2(x) = \cos x$ (Table 3).

Conclusion—We have presented a block method approach based on simultaneous finite difference methods which are embedded in a continuous k-step Adams-Moulton Method. The method eliminates the limitations associated with the Standard Adams-Moulton Method highlighted earlier and therefore leads to the following improvements: Once $y_1, \ldots, y_N$ are obtained, then $\bar{y}$ provides, from these discrete values, dense output, using (1.4). This is more economical than the discrete form. The method is self-starting and leads to superior uniform accuracy in $a \leq x \leq b$. (Tables 2 and 3).

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