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APPLICATION OF THE ENERGY METHOD FOR THE ANALYSIS OF BEAMS RESTING ON ELASTIC FOUNDATIONS

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Abstract

An approximate method of analysis for beams with free ends resting on an elastic soil medium is developed by using the energy method. Prismatic beams which are subjected to concentrated and uniformly distributed loads and concentrated moments are considered. The soil is idealized as a Winkler medium. Polynomial functions are used to define the deflected shape of the beam. Numerical results are developed to illustrate the influence of the relative rigidity of the beam-soil system on the deflections, flexural moments and shear forces of the beam.

Introduction

In the case of a deformed beam, Winkler's hypothesis (Winkler, 1867) leads to a fourth order linear differential equation and solutions have been obtained for a variety of loading and boundary conditions. These are available in several comprehensive books on the subject (Hayashi, 1921; Hetenyi, 1946; Vlasov and Leontiev, 1960). These solutions for the beam problem are either in a trigonometric or exponential series form or in the form of an infinite series, from which the evaluation of beam deflections or moments is generally very difficult. Various researchers have attempted to present the existing analytical solutions in a more usable form. Certain characteristic functions related to the normal modes of vibration of a beam have been used by Hendry (1958) and later by Iyenger and Anantharamu (1965) to examine beams on Winkler media. The relaxation method has been used by Wright (1952), matrix methods have been utilized by Mozingo (1967) and Bowles (1977), and the method of initial conditions has been used by Miranda and Nair (1966). Iwanczewska and Lewandowski (1968) have presented a wide range of slope-deflection equations for beams on a Winkler medium. Baker (1957) has developed an approximate analysis known as soil line method for the analysis of beams and strip footings resting on Winkler media. Kramrisch and Rogers (1961) have presented an approximate solution for beams and combined footings conforming to certain limitations regarding continuity, rigidity and variations in column loadings and spacings. The finite element method has been used by Just, et al. (1971) and the method of finite difference has been employed by Collatz (1960). By using the invariant imbedding technique, Distefano (1974) has developed an approximate method of analysis for beams resting on Winkler media.

In this research effort, ordinary polynomial functions are used to define the deflected shape of the beam on a Winkler medium. Solution of the problem is obtained by employing the principle of minimum energy of the beam-soil and load system.

Formulation

Consider a beam of width B and length L which is supported by a Winkler medium having a modulus of subgrade reaction equal to k. The material and cross-sectional properties of the beam are completely defined by the modulus of elasticity E and the moment of inertia I, respectively. It is assumed that the width of the beam is sufficiently small as compared to its length, so that only in-plane deformation of the beam will be considered. Let the beam be acted upon by a concentrated load (P), a concentrated moment (M) and a uniformly distributed load (of intensity q) as shown in Figure 1.

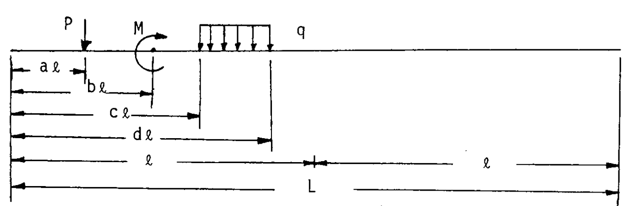

Fig. 1 A typical beam on Winkler medium.

In order to simplify the analysis, each load is divided into its symmetrical and antisymmetrical components as shown in Figure 2. Complete solution for any loading is obtained by superposition of the results from the corresponding symmeterical and antisymmetrical components. Based on the properties of the beam and soil medium, the nondimensional parameter, (λL), termed as relative rigidity is defined as:

$$\lambda L = \left(\frac{kb}{4EI}\right)^{1/4} L \tag{1}$$

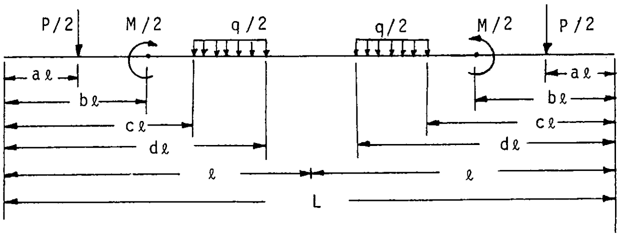

Fig. 2a Symmetrical load components for Fig. 1.

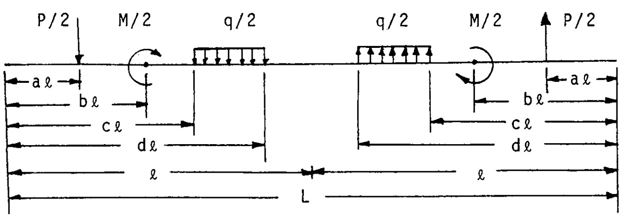

Fig. 2b Antisymmetrical load components for Fig. 1

SYMMETRIC LOADS

Considering the nature of the deflected shape of the beam for a symmetric loading, the displacement function will be taken as a polynomial of the form:

$$y = \ell\left[C_0 + \sum_{n=1}^{m} C_{2n}\left(\frac{x}{\ell}\right)^{2n}\right] \tag{2}$$

where $x = 0$ at the center of the beam, y is positive downward and $C_0, C_2, \ldots, C_{2m}$ are constants to be determined. The value of m is arbitrary and will be varied to study the accuracy of the solution as the number of terms of the polynomial is increased. The constant C_0 is eliminated by considering the equilibrium of the beam due to vertical forces. This condition would require that:

$$P + q\ell(d - c) - Bk\int_{-\ell}^{\ell} y\,dx = 0 \tag{3}$$

Substituting the expression for y from Equation 2 into Equation 3 and upon integration, it is found that

$$C_0 = \frac{P + q\ell(d - c)}{2kB\ell^2} - \sum_{n=1}^{m}\left(\frac{1}{2n+1}\right)C_{2n} \tag{4}$$

Therefore, the displacement function for the case of symmetric loads can be expressed in the following form:

$$y = \frac{P + q\ell(d - c)}{2kB\ell} + \ell\sum_{n=1}^{m} C_{2n}\left[\left(\frac{x}{\ell}\right)^{2n} - \frac{1}{2n+1}\right] \tag{5}$$

Having obtained the displacement function, one can calculate the total energy functional (Π) of the beam-soil and the loading system. The total energy functional is composed of the following components (Tauchert, 1974):

i) the strain energy due to the flexure of the beam (U_B);

ii) the strain energy due to the deformation of the idealized soil medium (U_F); and

iii) the potential energy of the externally applied loads (U_L)

so that:

$$U_B = \frac{EI}{2}\int_{-\ell}^{\ell}\left(\frac{d^2y}{dx^2}\right)^2 dx \tag{6a}$$

$$U_F = \frac{Bk}{2}\int_{-\ell}^{\ell} y^2\,dx \tag{6b}$$

$$U_L = -P[y]_{x=a\ell} - q\int_{c\ell}^{d\ell} y\,dx - M[dy/dx]_{x=b\ell} \tag{6c}$$

where the displacement function y is given by Equation 5 and the total energy functional Π is given by:

$$\Pi = U_B + U_F + U_L \tag{6d}$$

By substitution of the expression for y together with its derivatives dy/dx and d^2y/dx^2 from Equation 5, Equations (6a–c) can be integrated. The theorem of minimum potential energy states that of all kinematically admissible states of deformation those which satisfy equlibrium give a minimum of total energy, i.e., for the total energy functional Π to be minimum (Tauchert, 1974; Washizu, 1982):

$$\partial\Pi = 0, \quad \text{or}$$

$$\frac{\partial U_B}{\partial C_{2n}}dC_{2n} + \frac{\partial U_F}{\partial C_{2n}}dC_{2n} + \frac{\partial U_L}{\partial C_{2n}}dC_{2n} = 0 \tag{7}$$

The above condition generates the following set of simultaneous equations for the unknown constants C_{2n} ($n = 1, \ldots, m$):

$$\begin{aligned}\frac{8EI}{\ell}n(2n-1)\sum_{i=1}^{m}\frac{i(2i-1)}{2n+2i-3}C_{2i} \\ + 2Bk\ell^3\sum_{i=1}^{m} C_{2i} \times \\ \times\left[\frac{1}{2n+2i+1} - \frac{1}{(2n+1)(2i+1)}\right] \\ = P\ell\left(a^{2n} - \frac{1}{2n+1}\right) + 2Mnb^{2n-1} \\ + \frac{q\ell^2}{2n+1}\left(d^{2n+1} - c^{2n+1} + c - d\right)\end{aligned} \tag{8}$$

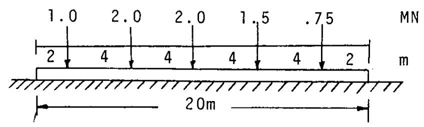

Fig.3 Data for the beam problem solved by Just, et al. (1971). Parameter values: k = 42.2 MN m^{-2}, EI = 1,675 MN m^{-2}, λL = 4.

where:

$$n = 1, 2, \ldots, m \tag{9}$$

ANTISYMMETRIC LOADS

For the loads shown in Figure 2b, the assumed displacement function is given by

$$y = \ell \sum_{n=1}^{m} C_{2n-1} \left(\frac{x}{\ell}\right)^{2n-1} \tag{10}$$

where C_1, C_3, ..., C_{2m-1} are constants to be determined.

Equation 10 is directly used to calculate the components U_B, U_F and U_L of the total energy functional. This is done by substituting Equation 10 into Equations (6a–d). On minimization of the total energy functional Π with respect to constants C_{2n-1} (n = 1, ..., m) as outlined by Equation 7, a set of simultaneous equations is derived. The constants C_{2n-1} are to be obtained from the following:

$$\begin{aligned} &\frac{8EI}{\ell}(2n-1)(n-1)\sum_{i=1}^{m}\frac{(2i-1)(i-1)}{2n+2i-5}C_{2i-1} \\ &\quad + 2Bk\ell^3 \sum_{i=1}^{m} \frac{C_{2i-1}}{2n+2i-1} \\ &= P\ell a^{2n-1} + M(2n-1)b^{2n-2} \\ &\quad + q\ell^2\left(\frac{d^{2n}-c^{2n}}{2n}\right) \end{aligned} \tag{11}$$

where n is given by Equation 9.

RESULTS AND DISCUSSION

The numerical results obtained from the energy method of analysis for beams have been compared with the exact results. The exact results are obtained from a computer program (Sarwar, 1987) following the analytical solution for a beam on Winkler medium given by Hetenyi (1946). To check the numerical accuracy of the present work, a beam as shown in Figure 3 is analyzed. This particular problem was also examined by Just et al. (1971) by the finite element method.

The comparison of the beam displacements is presented in Table 1. The results indicate that the magnitude of error is very small. The distribution of flexural moments is graphically presented in Figure 4. It is observed that the results obtained from the energy method underestimate the values given by Just et al. (1971) as well as those results obtained from the exact analysis. This difference is particularly noticeable at the points directly below the loads, the maximum error being of the order of 4%. The comparison for shear forces is presented in Figure 5. Unlike flexural moments, computed values for shearing forces are overestimated by approximately 8% at the center of the beam.

Table 1 Comparison of Displacements

Dist. from left edge (m)	Exact value (m × 10^{-3})	Energy solution (m × 10^{-3})	Percent error
0.0	4.10539	4.06106	−1.08
2.0	6.84542	6.81051	−0.51
4.0	9.14745	9.19227	0.49
6.0	11.15550	11.08746	−0.62
8.0	11.76190	11.85717	0.81
10.0	11.78610	11.68827	−0.83
12.0	10.56230	10.64891	0.82
14.0	9.09365	9.01999	−0.81
16.0	6.95958	6.99786	0.55
18.0	4.87365	4.84294	−0.63
20.0	2.55399	2.52437	−1.16

Since in the energy method only a finite number of terms of the polynomial function can be evaluated to define the deflected shape of a beam, the solution will always be approximate. The degree of accuracy of a given solution will, therefore, be dependent on the number of terms used for a particular analysis. A beam with the loading and material properties as shown in Figure 6 is used to investigate the accuracy of various approximations. The results are compared with the exact values in Table 2. It is observed that with the use of an increased number of terms, the results are improved. For the particular problem, while the convergence in displacements is obtained with the use of only 10 terms, accurate results for flexural moments and shear forces would require the use of larger numbers of terms. If only four terms are used, the maximum variation in moment is approximately 4% less than the exact value. It may be worth pointing out that the moment and shear are less accurate because the moment is the second derivative of the assumed deflection function and

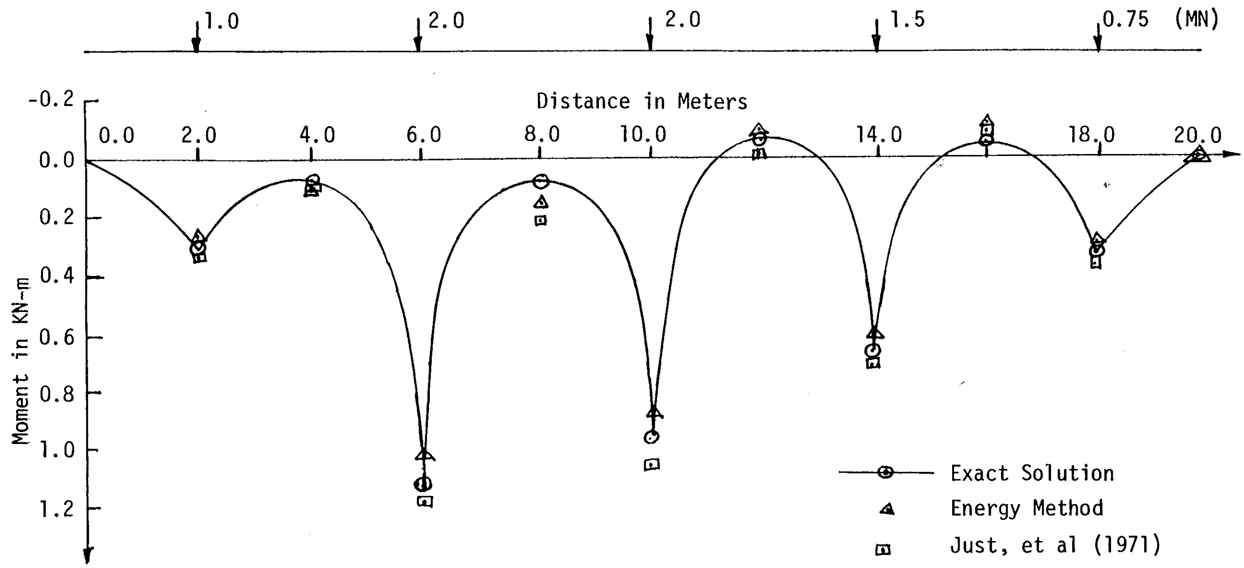

Fig. 4 Comparison of moments (based on $\lambda L = 4$).

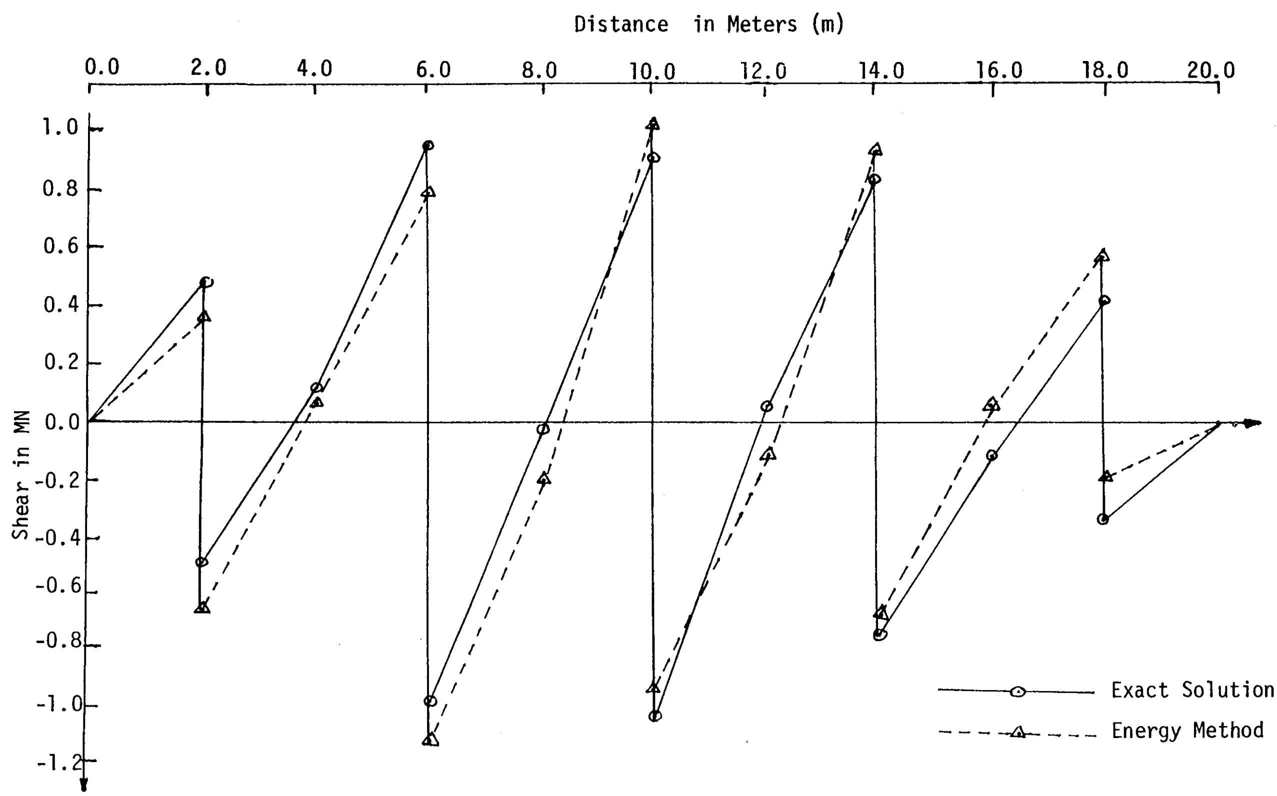

Fig. 5 Comparison of shear (based on $\lambda L = 5.6$).

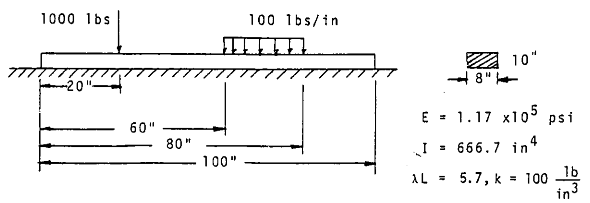

Fig. 6 Beam loading for the study of convergence.

Table 2 Convergence of Polynomial Solution (Based on Fig. 6)

Number of terms in the polynomial function	Percent error with respect to exact solution
Displacement	
2	−2.498
4	−0.126
6	−0.020
8	0.008
10	0.001
Moment	
2	−20.538
4	−3.245
6	−0.599
8	0.827
10	0.329
Shear	
2	−76.745
4	19.338
6	16.445
8	5.459
10	0.402

shear is a third derivative.

The results presented so far are based on the relative rigidity parameter, λL, in the range of 4 to 6. For rigid beams with low values of λL (say less than 3), better agreement exists between the computed and exact values. As λL increases, the error in the computed values increases. To study this factor, a beam with a concentrated load at its center is analyzed for a wide range of λL, ranging from 1 to 10. The computed values are compared in Table 3. From the results for displacements it is observed that the energy solution yields approximately the same result as the exact method. But for flexural moments and shear forces, the magnitude of error increases as λL increases. Also the error in the calculation for shear is higher than that for flexural moment. At $\lambda L = 10$, the maximum error in shear being approximately one half of the exact value. This suggests that for very flexible beams (such as with $\lambda L \geq 10$), the results for shear might be highly inaccurate. Fortunately for us, since all beams encountered in practice have the values of relative rigidity λL in the range from 1 to 6, the results of the energy solution can be used as a useful tool for design purposes.

Table 3 Effects of Relative Rigidity (λL)

λL	Exact method	Energy method
	Displacements at center/(P/kL)	
1	1.0124	1.0124
4	2.1599	2.1581
10	5.0008	4.9298
	Moments at center/(PL)	
1	1.2143×10^{-1}	1.1614×10^{-1}
4	6.5866×10^{-2}	5.7697×10^{-2}
10	2.5003×10^{-2}	1.6868×10^{-2}
	Shear (at $x = 0.4L$)/P	
1	3.9883×10^{-1}	4.6558×10^{-1}
4	2.9330×10^{-1}	3.6002×10^{-1}
10	9.9329×10^{-2}	1.6501×10^{-1}

Conclusions

For beams resting on Winkler media, solution by the energy method is developed and the convergence of the solution to the exact results is established. Numerical results are developed to illustrate the influence of relative rigidity of the beam-soil-load system on beam deflections, flexural moments and shear forces. It is observed that for beams having low values of relative rigidities ($\lambda L \leq 4$), the energy solution yields very satisfactory results. The results for flexural moments show only slight variation (of the order of 4% and 8% below the exact solution, respectively). The effect of increased relative rigidity of the beam-soil system is to increase the deflection of the beam at or near the point of load application. The flexural moments and shear forces are, however, decreased. The present solution responds to increased λL favorably well, provided that $\lambda L \leq 6$. Based on practical standpoints, the energy solution obtained from the use of only 4 terms can be used to adequately determine the displacements, moments and shear forces in a beam.

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