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NUMERICAL TRANSFORM METHODS FOR SOLVING SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper presents a technique that uses both Laplace and Fourier transforms to solve a system of partial differential equations. Numerical method is also used to relate the frequency domain to time domain of the solution. A system of four variables is selected as an example to illustrate the technique.

INTRODUCTION

In this paper the transform methods refer to Laplace and Fourier transforms. The Laplace transform reduces the solution of a linear total or partial differential equation to essentially an algebraic procedure. In addition, the relevant boundary conditions are introduced early in the analysis, and the constants of integration are automatically evaluated. The final solution of the equation is obtained by the inverse Laplace transform However, Laplace transform is not always ascertained for the solution of the equation resulting from certain complicated situations. This difficulty can be overcome by applying Fourier transform which converts a complex variable into time domain. Therefore, the two transforms can be used to solve a system of partial differential equations when the inverse Laplace transform is not feasible for the solution.

DEFINITIONS OF THE TRANSFORMS

The Laplace transform of a function f(t) is defined for positive value of t as a function of new variables by the integral

$$\mathcal{L}[f(t)] = \int_0^{\infty} e^{-st} f(t)\,dt$$

The transform exists if f(t) satisfies the following conditions:

1. f(t) is continuous or piecewise continuous in any interval $t_1 \leq t \leq t_2$, where $t_1 > 0$.
2. $t^n \mid f(t) \mid$ is bounded near $t = 0$ when approached from positive values of t for some number n, when $n < 1$.
3. The $e^{-S_0 t} \mid f(t) \mid$ is bounded for large values of t for some number s_0.

The function f(t) is said to be piecewise continuous in the range $t_1 \leq t \leq t_2$ if it is possible to divide the range into a finite number of intervals in such a way that f(t) is continuous within each interval and approaches finite values as either end of the intervals is approached from within the interval. Thus, a piecewise continuous function may have a number of discontinuities; and the method may be used for the solution of a finite difference equation as well as a differential equation.

In dealing with ordinary functions, Laplace transform is more favorable because Fourier transforms of many functions occuring in practice do not exist. However, we may find that they do exist in a generalized sense. Furthermore, since the Fourier transform is simpler conceptually and is more meaningful physically than the Laplace transform, we shall find that an operational calculus for generalized functions based on the Fourier transform is a fairly ideal one. For practical application, a restricted definition is given as follows:

If f(t) is a real function of time which is zero for all $t < 0$, then its associated Fourier transform pair (Guillemin, 1963) is

$$F(jw) = \int_0^{\infty} f(t) e^{-jwt} dt \;, \quad f(t) = 0 \;, \; t < 0 \qquad (1)$$

$$f(t) = \int_0^{\infty} F(jw) e^{-jwt} dw \;, \quad f(t) = \text{real}, \quad j^2 = -1 \qquad (2)$$

NUMERICAL TRANSFORMS

Equation (1) may be broken into real and imaginary parts and evaluated with special quadrature formulas as shown below:

$$F(jw) = \int_0^{\infty} f(t) \text{Cos} wt\,dt - j\int_0^{\infty} f(t) \text{Sin} wt\,dt \qquad (3)$$

or

$$F(jw) = R(w) + jI(w) \qquad (4)$$

$$\cong \Delta t \sum_{k=1}^{n} f(k\Delta t)\text{Cos}(wk\Delta t) - j\Delta t \sum_{k=1}^{n} f(k\Delta t)\text{Sin}(wk\Delta t)$$

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However, the following method developed by Harris, et al. (1967) is more flexible and efficient if f(t) can be represented by a function made up of a series of polynomial segments. Ordinary discontinuity in the function and its derivatives are allowed at the union of the segments. The time domain for this function is

$$f(t) \cong \sum_{i=1}^{n} u(t - T_i)(A_i + B_i(t - T_i) + C_i(t - T_i)^2 + \ldots.) \quad (5)$$

where A_i is the change in the function over the interval T_i-1 to T_i, B_i is the change in the first derivative at T_i over the interval, and the change in the second derivative at T_i is C_i, etc. The notation $u(t - T_i)$ represents a unit step function and it indicates that additions to the function f(t), causing the jumps at T_i, occur at time T_i. This expression is quite flexible and the approximating function can be made as simple as desired or as complex as desired for any individual case. It includes stepwise, linear and parabolic approximations to f(t) as special cases.

The Laplace transformation of Equation (5) is

$$F(S) \cong \sum_{i=1}^{n} \left[\frac{A_i}{S} + \frac{B_i}{S^2} + \frac{2C_i}{S^3} + \frac{6D_i}{S^4} + \ldots. \right] e^{-T_i S} \quad (6)$$

If the function f(t) is Fourier transformable, jw can be substituted for S to give the Fourier transform

for S to give the Fourier transform

$$F(jw) \equiv \sum_{i=1}^{n} \left(\frac{A_i}{jw} + \frac{B_i}{-w^2} + \frac{2C_i}{-jw^3} + \frac{6D_i}{w^4} + \ldots. \right) e^{-jwT_i} \quad (7)$$

$$\equiv \sum_{i=1}^{n} \left[\left(-\frac{A_i}{w} + \frac{2C_i}{w^3} - \ldots.\right) \mathrm{Sin}\, wT_i + \left(\frac{B_i}{w^2} + \frac{6D_i}{w^4} - \ldots\right) \mathrm{Cos}\, wT_i \right]$$

$$+ j \sum_{i=1}^{n} \left[\left(-\frac{A_i}{w} + \frac{2C_i}{w^2} - \ldots\right) \mathrm{Cos}\, wT_i - \left(-\frac{B_i}{w^2} + \frac{6D_i}{w^4} - \ldots\right) \mathrm{Sin}\, wT_i \right]$$

In terms of its real and imaginary parts, it can be written as

$$R(w) \equiv \sum_{i=1}^{n} \left[\left(-\frac{A_i}{w} + \frac{2C_i}{w^3} - \ldots\right) \mathrm{Sin}\, wT_i + \left(-\frac{B_i}{w^2} + \frac{6D_i}{w^4} - \ldots.\right) \mathrm{Cos}\, wT_i \right] \quad (8)$$

$$I(w) \equiv \sum_{i=1}^{n} \left[\left(-\frac{A_i}{w} + \frac{2C_i}{w^3} - \ldots\right) \mathrm{Cos}\, wT_i - \left(-\frac{B_i}{w^2} + \frac{6D_i}{w^4} - \ldots.\right) \mathrm{Sin}\, wT_i \right] \quad (9)$$

The symmetry of the Fourier transform pair makes possible the use of Equations (8) and (9) for inverse numerical Fourier transformation as well as for evaluation of the direct Fourier transform. Since f(t) is zero for negative t, Equation (2) can be written as

$$f(t) = \frac{2}{\pi} \int_0^\infty R(w) \mathrm{Cos}\, wt\, dw \quad (10)$$

and

$$f(t) = -\frac{2}{\pi} \int_0^\infty I(w) \mathrm{Sin}\, wt\, dw \quad (11)$$

These integrals, which express the inverse Fourier transform, are exactly the same as the integrals that express the direct transform, as given in Equation (3), except that t and w are interchanged. R(w) and I(w) can thus be approximated by polynomial segments, and Equations (8) and (9) can be used to compute the integrals for the inverse transformation in the following manner:

$$R(w) \equiv \sum_{i=1}^{n} u(w - W_i) \left[A_i' + B_i'(w - W_i) + C_i'(w - W_i)^2 + D_i'(w - W_i)^3 + \ldots \right] \quad (12)$$

where A_i', B_i', etc., refer to R(w) and its derivatives. Therefore,

$$f(t) \equiv \frac{2}{\pi} \sum_{i=1}^{n} \left[\left(-\frac{A_i'}{t} + \frac{2C_i'}{t^3} - \ldots\right) \mathrm{Sin}\, tW_i + \left(-\frac{B_i'}{t^2} + \frac{6D_i'}{t^4} - \ldots\right) \mathrm{Cos}\, tW_i \right] \quad (13)$$

In like manner I(w) can be written as

$$I(w) \equiv \sum_{i=1}^{n} u(w - W_i) \left[A_i' + B_i'(w - W_i) + C_i'(w - W_i)^2 + D_i'(w - W_i)^3 + \ldots \right] \quad (14)$$

and f(t) is given by

$$f(t) \equiv -\frac{2}{\pi} \sum_{i=1}^{n} \left[\left(-\frac{A_i'}{t} + \frac{2C_i'}{t^3} - \ldots\right) \mathrm{Cos}\, tW_i - \left(-\frac{B_i'}{t^2} + \frac{6D_i'}{t^4} - \ldots.\right) \mathrm{Sin}\, tW_i \right] \quad (15)$$

In conclusion, it is better to use Equation (15) for functions that have no discontinuity at the origin, while Equation (13) is more suitable for inversion of functions with such a discontinuity. Clements, et al. (1963) have discussed the relative advantages of using the two equations.

AN EXAMPLE

As an example of using the transforms for the solution of differential equations, consider the following models:

$$\frac{\partial y}{\partial t} = k(z - y) \quad (16)$$

$$\frac{\partial z}{\partial t} = a \frac{\partial^2 z}{\partial x^2} - b \frac{\partial z}{\partial x} + c(y - z) \quad (17)$$

where a, b, c, k are the model parameters, while x, y, z, t the system variables. The solution of z in time domain is desired. The following boundary conditions apply,

$z(x,0) = 0$	for $x > 0$
$y(x,0) = 0$	
$\lim_{x \to \infty} z(x,t) = 0$	for $t \geq 0$
$z(0,t) = z_0$	finite value

Taking Laplace transforms of Equations (16) and (17) and using the boundary conditions give

$$S\bar{y}(x,S) - 0 = k(\bar{z}(x,S) - \bar{y}(x,S)) \quad (18)$$

$$S\bar{z}(x,S) - 0 = a \frac{d^2\bar{z}(x,S)}{dx^2} - b \frac{d\bar{z}}{dx} + c(\bar{y}(x,S) - \bar{z}(x,S)) \quad (19)$$

From Equation (18), Y(x,S) is obtained and substituted in Equation (19). Then the model becomes an ordinary transformed differential equation,

$$a \frac{d^2\bar{z}}{dx^2} - b \frac{d\bar{z}}{dx} + \left(\frac{kc}{S + k} - c - S\right)\bar{z} = 0 \quad (20)$$

The solution of Equation (20) is given by Lai (1966):

$$\bar{z}(x,S) = B_1 \exp\left[\frac{b + \sqrt{b^2 - 4aq}}{2a}\right] x + B_2 \exp\left(\frac{b - \sqrt{b^2 - 4aq}}{2a}\right) x \quad (21)$$

where $q = (kc/S + k) - c - S$, and B_1 and B_2 are integration constants determined by the boundary conditions which give $B_1 = 0$, $B_2 = z_0$. The final solution of the equation is

$$\bar{z}(x,S) = z_0 \exp\left(\frac{b - \sqrt{b^2 - 4aq}}{2a}\right) x \quad (22)$$

The function z(t) is assumed to be Fourier transformable, and so jw can be substituted for S to give the Fourier transform at a specified values of x = L and applying de Moivre's theorem, the expression z(jw) becomes

$$z(jw) = z_0 \exp\left[x\left(\frac{b - r^{\frac{1}{2}} \mathrm{Cos} \frac{1}{2}\theta}{2a}\right)\right] \mathrm{Cos}\,\beta - j z_0 \exp\left[x\left(\frac{b - r^{\frac{1}{2}} \mathrm{Cos} \frac{1}{2}\theta}{2a}\right)\right] \mathrm{Sin}\,\beta$$

where

$$r^{\frac{1}{2}} = (m^2 + n^2)^{\frac{1}{2}}, \quad m = \frac{b^2(k^2 + w^2) + 4acw^2}{k^2 + w^2}$$

$$\theta = \mathrm{Tan}^{-1} n/m$$

$$\beta = \frac{r^{\frac{1}{2}} \mathrm{Sin} \frac{1}{2}\theta}{2}, \quad n = \frac{4a(w^3 + ckw + k^2 w)}{k^2 + w^2}$$

For computational purposes, the function is converted into real, R(w), and imaginary, I(w), parts :

$$R(w) = z_0 \exp\left[x\left(\frac{b - r^{\frac{1}{2}} \mathrm{Cos} \frac{1}{2}\theta}{2a}\right)\right] \mathrm{Cos}\,\beta$$

and

$$I(w) = z_0 \exp\left[x\left(\frac{b - r^{\frac{1}{2}} \mathrm{Cos} \frac{1}{2}\theta}{2a}\right)\right] \mathrm{Sin}\,\beta$$

Applying Equations (12) through (15), gives the numerical solution

$$z(t) \equiv \frac{2}{\pi} \sum_{i=1}^{n} \left[\left(-\frac{\Delta R_i}{t} + \frac{2\Delta R_i''}{t^3} - \ldots\right) \mathrm{Sin}\, tW_i + \left(-\frac{\Delta R_i'}{t^2} + \frac{6\Delta R_i'''}{t^4} - \ldots\right) \mathrm{Cos}\, tW_i \right] \quad (23)$$

or

$$z(t) \equiv -\frac{2}{\pi} \sum_{i=1}^{n} \left[\left(-\frac{\Delta R_i}{t} + \frac{2\Delta R_i''}{t^3} - \ldots\right) \mathrm{Cos}\, tW_i - \left(-\frac{\Delta R_i'}{t^2} + \frac{6\Delta R'''}{t^4} - \ldots.\right) \mathrm{Sin}\, tW_i \right] \quad (24)$$

in which ΔR_i, $\Delta R_i'$, $\Delta R_i''$, $\Delta R_i'''$ etc., refer to R(w) in Equation (23), and to I(w) in Equaiton (24), and their derivatives.

CONCLUSIONS

In the solution of a differential equation by classical method, we must determine two solutions, the general solution of the homogeneous equation and a particular solution of the non-homogeneous equation, using different techniques. Furthermore, the initial conditions are taken into account only after we have determined the general solution of the differential equation, and the arbitrary constants are determined so that the solution satisfies these conditions. With the Laplace transform method we may obtain the solution to differential equations without having to distinguish between parts of the solution. Furthermore, with the initial conditions built in, the Laplace method is particularly useful when applied to certain system of differential equations. In the classical method for solving system of equations we are confronted with the problem as to whether the general solution involves more constants than the essential number of constants; Laplace transform method does not present this problem. However, the inverse Laplace transform is not always readily ascertained. This difficulty could be overcome by numerical Fourier Transform pair as shown in the example.

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