A RESTUDY OF THE DERIVATIVE OF RATIONAL POWERS OF A FUNCTION

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Some authors of textbooks on the calculus develop the formula for the derivative of a positive integral power by use of the binomial theorem and then assume the result is valid for any rational power. Other authors, after making use of the binomial theorem to obtain the derivative of a positive integral power, use implicit differentiation to obtain the required formula for all other rational powers of the function.

It is the purpose of this note to show that the derivative of a rational power of a function may be obtained without the use of either method mentioned in the first paragraph, by giving a method which yields the result for all possible rational powers.

Let
$$y = u^n$$
where $u = f(x)$ and n is a rational number. There are four cases to be considered.

Case 1. Let n be a positive integer.

If x takes on the increment $\triangle\, x$, y and u assume the values $y + \triangle\, y$ and $u + \triangle\, u$. Consequently
$$\triangle\, y = (u + \triangle\, u)^n - u^n. \tag{1}$$
When the right side of (1) is factored
$$\triangle\, y = H \triangle\, u \tag{2}$$
where
$H = (u + \triangle\, u)^{n-1} + (u + \triangle\, u)^{n-2}\, u + \ldots + (u + \triangle\, u)u^{n-2} + u^{n-1}$.

Upon dividing both sides of (2) by $\triangle\, x$ the resulting equation
$$\frac{\triangle\, y}{\triangle\, x} = H\,\frac{\triangle\, u}{\triangle\, x} \tag{3}$$
follows. Since $\triangle\, u \to 0$ as $\triangle\, x \to 0$, $\lim\limits_{\triangle x \to 0} H = nu^{n-1}$. On passing to the limit as $\triangle\, x \to 0$, (3) becomes
$$\frac{dy}{dx} = nu^{n-1}\,\frac{du}{dx}, \tag{4}$$
which is the usual result for the positive integer n.

Case 2. Let n be a negative integer.

Write $n = -m$. Then
$$y = u^{-m} = \frac{1}{u^m}.$$

Consequently

$$\triangle y = \frac{u^m - (u + \triangle u)^m}{u^m (u + \triangle u)^m} \tag{5}$$

$$= - \frac{H \triangle u}{u^m (u + \triangle u)^m} \tag{6}$$

where

$H = [u^{m-1} + u^{m-2} (u + \triangle u) + \ldots + u(u + \triangle u)^{m-2} + (u + \triangle u)^{m-1}]$.

Now $\lim_{\triangle x \to 0} H = mu^{m-1}$ and $\lim_{\triangle x \to 0} (u + \triangle u)^m = u^m$. Thus when both sides of (6) are divided by $\triangle x$ and the limit as $\triangle x \to 0$ is taken

$$\frac{dy}{dx} = - \frac{mu^{m-1} \frac{du}{dx}}{u^{2m}}$$

$$\frac{dy}{dx} = - mu^{-m-1} \frac{du}{dx}$$

or

$$\frac{dy}{dx} = nu^{n-1} \frac{du}{dx} \tag{7}$$

since $n = -m$. The formula is therefore obtained if n is a negative integer.

Case 3. Let n be a positive fraction.

Write $n = p/q$ where p/q is a fraction in its lowest terms.

Then

$$\triangle y = (u + \triangle u)^{p/q} - u^{p/q}. \tag{8}$$

If the numerator and denominator of (8) are multiplied by G, where

$$G = [(u + \triangle u)^{p/q\ (q-1)} + (u + \triangle u)^{p/q\ (q-2)} u + \ldots + (u + \triangle u) u^{p/q\ (q-2)} + u^{p/q\ (q-1)}]$$

then

$$\triangle y = \frac{(u + \triangle u)^p - u^p}{G}.$$

This result may be written

$$\triangle y = \frac{H}{G} \triangle u \tag{9}$$

where

$$H = [(u + \triangle u)^{p-1} + (u + \triangle u)^{p-2} u + \dots + (u + \triangle u) u^{p-2} + u^{p-1}].$$

Now $\lim_{\triangle x \to 0} G = q u^{p/q\ (q-1)}$ and $\lim_{\triangle x \to 0} H = p u^{p-1}$.

Upon dividing both sides of (9) by $\triangle x$, and obtaining the limit as $\triangle x \to 0$, it follows that

$$\frac{dy}{dx} = \frac{p u^{p-1} \frac{du}{dx}}{q u^{p/q\ (q-1)}}$$

or

$$\frac{dy}{dx} = \frac{p}{q} u^{p/q - 1} \frac{du}{dx} \tag{10}$$

which is the correct result for $n = p/q$.

Case 4. Let n be a negative fraction.

Put $n = -p/q$, where p and q are positive relatively prime integers. Then

$$\triangle y = \frac{1}{(u + \triangle u)^{p/q}} - \frac{1}{u^{p/q}}$$

or

$$\triangle y = \frac{u^{p/q} - (u + \triangle u)^{p/q}}{u^{p/q} (u + \triangle u)^{p/q}}. \tag{11}$$

If both numerator and denominator of (11) are multiplied by G, where

$$G = [u^{p/q\ (q-1)} + u^{p/q\ (q-2)} (u + \triangle u)^{p/q} + \dots + u^{p/q} (u + \triangle u)^{p/q\ (q-2)} + (u + \triangle u)^{p/q\ (q-1)}]$$

(11) becomes

$$\triangle y = \frac{u^p - (u + \triangle u)^p}{u^{p/q} (u + \triangle u)^{p/q} G},$$

or

$$\triangle y = -\frac{H \triangle u}{u^{p/q} (u + \triangle u)^{p/q} G} \tag{12}$$

in which

$$H = [u^{p-1} + u^{p-2} (u + \triangle u) + \dots + u (u + \triangle u)^{p-2} + (u + \triangle u)^{p-1}].$$

But lim $H = pu^{p-1}$, lim $G = qu^{p/q\ (q-1)}$ and lim $(u + \triangle\ u)^{p/q} = u^{p/q}$.
$\triangle\ x \rightarrow 0$ $\triangle\ x \rightarrow 0$ $\triangle\ x \rightarrow 0$

Consequently, if both sides of (12) are divided by $\triangle\ x$, and the limit as $\triangle\ x \rightarrow 0$ is obtained

$$\frac{dy}{dx} = - \frac{pu^{p-1}\frac{du}{dx}}{u^{p/q}\ u^{p/q}\ [qu^{p/q\ (q-1)}]}$$

or

$$\frac{dy}{dx} = - \frac{p}{q}\ u^{-p/q-1}\ \frac{du}{dx} \qquad (13)$$

which is the result for $n = {}^{-p/q}$. Thus the formula for the derivative of a power of a function is obtained, subject only to the condition that the power be rational.

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